

CLTI Differential Equation

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Causal LTI Systems (1)

$$a_N \frac{d^N y(t)}{d t^N} + a_{N-1} \frac{d^{N-1} y(t)}{d t^{N-1}} + \cdots + a_1 \frac{d y(t)}{d t} + a_0 y(t) = b_M \frac{d^M x(t)}{d t^M} + b_{M-1} \frac{d^{M-1} x(t)}{d t^{M-1}} + \cdots + b_1 \frac{d x(t)}{d t} + b_0 x(t)$$

$$\begin{aligned} \frac{d^N y(t)}{d t^N} + a_1 \frac{d^{N-1} y(t)}{d t^{N-1}} + \cdots + a_{N-1} \frac{d y(t)}{d t} + a_N y(t) &= b_{N-M} \frac{d^M x(t)}{d t^M} + b_{N-M+1} \frac{d^M x(t)}{d t^{M-1}} + \cdots + b_{N-1} \frac{d x(t)}{d t} + b_N x(t) \\ (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) &= (D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) x(t) \\ Q(D) y(t) &= P(D) x(t) \end{aligned}$$

$$M = N$$

$$\begin{aligned} \frac{d^N y(t)}{d t^N} + a_1 \frac{d^{N-1} y(t)}{d t^{N-1}} + \cdots + a_{N-1} \frac{d y(t)}{d t} + a_N y(t) &= b_0 \frac{d^M x(t)}{d t^M} + b_1 \frac{d^M x(t)}{d t^{M-1}} + \cdots + b_{N-1} \frac{d x(t)}{d t} + b_N x(t) \\ (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) &= (b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N) x(t) \\ Q(D) y(t) &= P(D) x(t) \end{aligned}$$

Causal LTI Systems (2)

$$a_N \frac{d^N y(t)}{d t^N} + a_{N-1} \frac{d^{N-1} y(t)}{d t^{N-1}} + \cdots + a_1 \frac{d y(t)}{d t} + a_0 y(t) = b_M \frac{d^M x(t)}{d t^M} + b_{M-1} \frac{d^{M-1} x(t)}{d t^{M-1}} + \cdots + b_1 \frac{d x(t)}{d t} + b_0 x(t)$$

$$\frac{d^N y(t)}{d t^N} + a_1 \frac{d^{N-1} y(t)}{d t^{N-1}} + \cdots + a_{N-1} \frac{d y(t)}{d t} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{d t^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{d t^{M-1}} + \cdots + b_{N-1} \frac{d x(t)}{d t} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

$$Q(D) y(t) = P(D) x(t)$$

- Zero Input Response
- Zero State Response (Convolution with $h(t)$)

- Natural Response (Homogeneous Solution)
- Forced Response (Particular Solution)

Zero Input Response $y_0(t) - (1)$

$$\frac{d^N y(t)}{d t^N} + \color{red}{a_1} \frac{d^{N-1} y(t)}{d t^{N-1}} + \cdots + \color{red}{a_{N-1}} \frac{d y(t)}{d t} + \color{red}{a_N} y(t) = \color{green}{b_{N-M}} \frac{d^M x(t)}{d t^M} + \color{green}{b_{N-M+1}} \frac{d^M x(t)}{d t^{M-1}} + \cdots + \color{green}{b_{N-1}} \frac{d x(t)}{d t} + \color{green}{b_N} x(t)$$

$$(\mathcal{D}^N + \color{red}{a_1} \mathcal{D}^{N-1} + \cdots + \color{red}{a_{N-1}} \mathcal{D} + \color{red}{a_N}) y(t) = (\mathcal{D}^M + \color{green}{b_{N-M+1}} \mathcal{D}^{M-1} + \cdots + \color{green}{b_{N-1}} \mathcal{D} + \color{green}{b_N}) x(t)$$

$$\mathcal{Q}(\mathcal{D}) y(t) = P(\mathcal{D}) x(t)$$

$$\mathcal{Q}(\mathcal{D}) y_0(t) = 0 \quad \Rightarrow \quad (\mathcal{D}^N + \color{red}{a_1} \mathcal{D}^{N-1} + \cdots + \color{red}{a_{N-1}} \mathcal{D} + \color{red}{a_N}) y_0(t) = 0$$



Linear combination of $y_0(t)$ and its derivatives = 0

if and only if

$$y_0(t) = ce^{\lambda t}$$

$$\dot{y}_0(t) = c\lambda e^{\lambda t}$$

$$\ddot{y}_0(t) = c\lambda^2 e^{\lambda t}$$

...

$$\mathcal{Q}(\lambda) = 0 \quad \Leftrightarrow \quad \frac{(\lambda^N + \color{red}{a_1} \lambda^{N-1} + \cdots + \color{red}{a_{N-1}} \lambda + \color{red}{a_N})}{= 0} \underline{ce^{\lambda t}} \neq 0$$

Zero Input Response $y_0(t)$ – (2)

$$\begin{aligned} \frac{d^N y(t)}{d t^N} + \color{red}{a_1} \frac{d^{N-1} y(t)}{d t^{N-1}} + \cdots + \color{red}{a_{N-1}} \frac{d y(t)}{d t} + \color{red}{a_N} y(t) &= \color{green}{b_{N-M}} \frac{d^M x(t)}{d t^M} + \color{green}{b_{N-M+1}} \frac{d^M x(t)}{d t^{M-1}} + \cdots + \color{green}{b_{N-1}} \frac{d x(t)}{d t} + \color{green}{b_N} x(t) \\ (\color{blue}{D}^N + \color{red}{a_1} \color{blue}{D}^{N-1} + \cdots + \color{red}{a_{N-1}} \color{blue}{D} + \color{red}{a_N}) y(t) &= (\color{blue}{D}^M + \color{green}{b_{N-M+1}} \color{blue}{D}^{M-1} + \cdots + \color{green}{b_{N-1}} \color{blue}{D} + \color{green}{b_N}) x(t) \\ Q(\color{blue}{D}) y(t) &= P(\color{blue}{D}) x(t) \end{aligned}$$

$$Q(\color{blue}{D}) y_0(t) = 0 \quad \Rightarrow \quad (\color{blue}{D}^N + \color{red}{a_1} \color{blue}{D}^{N-1} + \cdots + \color{red}{a_{N-1}} \color{blue}{D} + \color{red}{a_N}) y_0(t) = 0$$

$$Q(\lambda) = 0 \quad \iff \quad \frac{(\lambda^N + \color{red}{a_1} \lambda^{N-1} + \cdots + \color{red}{a_{N-1}} \lambda + \color{red}{a_N})}{= 0} ce^{\lambda t} \neq 0$$

$$Q(\lambda) = (\lambda^N + \color{red}{a_1} \lambda^{N-1} + \cdots + \color{red}{a_{N-1}} \lambda + \color{red}{a_N}) = 0$$

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_N) \quad \lambda_i \quad \text{characteristic roots}$$

$$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \cdots + c_N e^{\lambda_N t} = y_0(t) \quad e^{\lambda_i t} \quad \text{characteristic modes}$$

ZIR: a linear combination of the characteristic modes of the system

Zero State Response $y(t) - (1)$

$$\begin{aligned}\frac{d^N y(t)}{d t^N} + \color{red}{a_1} \frac{d^{N-1} y(t)}{d t^{N-1}} + \cdots + \color{red}{a_{N-1}} \frac{d y(t)}{d t} + \color{red}{a_N} y(t) &= \color{green}{b_{N-M}} \frac{d^M x(t)}{d t^M} + \color{green}{b_{N-M+1}} \frac{d^M x(t)}{d t^{M-1}} + \cdots + \color{green}{b_{N-1}} \frac{d x(t)}{d t} + \color{green}{b_N} x(t) \\ (\color{blue}{D}^N + \color{red}{a_1} \color{blue}{D}^{N-1} + \cdots + \color{red}{a_{N-1}} \color{blue}{D} + \color{red}{a_N}) y(t) &= (\color{blue}{D}^M + \color{green}{b_{N-M+1}} \color{blue}{D}^{M-1} + \cdots + \color{green}{b_{N-1}} \color{blue}{D} + \color{green}{b_N}) x(t) \\ Q(\color{blue}{D}) y(t) &= P(\color{blue}{D}) x(t)\end{aligned}$$

All initial conditions are zero

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) y(t - \tau) d\tau$$

Impulse response $h(t)$

$$y(t) = \int_{0^-}^{+t} x(\tau) y(t - \tau) d\tau , \quad t \geq 0$$

Causality

causal system: Response cannot begin before the input

causal input: The input starts at $t=0$ $h(\tau) = 0 \quad \tau < 0$

causal $h(t)$: The causal system's response to a unit impulse cannot begin before $t=0$

$$h(t - \tau) = 0 \quad t - \tau < 0$$

Total Response

$$\frac{d^N y(t)}{d t^N} + \color{red}{a_1} \frac{d^{N-1} y(t)}{d t^{N-1}} + \cdots + \color{red}{a_{N-1}} \frac{d y(t)}{d t} + \color{red}{a_N} y(t) = \color{green}{b_{N-M}} \frac{d^M x(t)}{d t^M} + \color{green}{b_{N-M+1}} \frac{d^M x(t)}{d t^{M-1}} + \cdots + \color{green}{b_{N-1}} \frac{d x(t)}{d t} + \color{green}{b_N} x(t)$$

$$(\mathcal{D}^N + \color{red}{a_1} \mathcal{D}^{N-1} + \cdots + \color{red}{a_{N-1}} \mathcal{D} + \color{red}{a_N}) y(t) = (\mathcal{D}^M + \color{green}{b_{N-M+1}} \mathcal{D}^{M-1} + \cdots + \color{green}{b_{N-1}} \mathcal{D} + \color{green}{b_N}) x(t)$$

$$Q(\mathcal{D}) y(t) = P(\mathcal{D}) x(t)$$

$$y(t) = \underbrace{\sum_{k=1}^N c_k e^{\lambda_k t}}_{\text{Zero Input Response}} + \underbrace{x(t) * h(t)}_{\text{Zero State Response}}$$

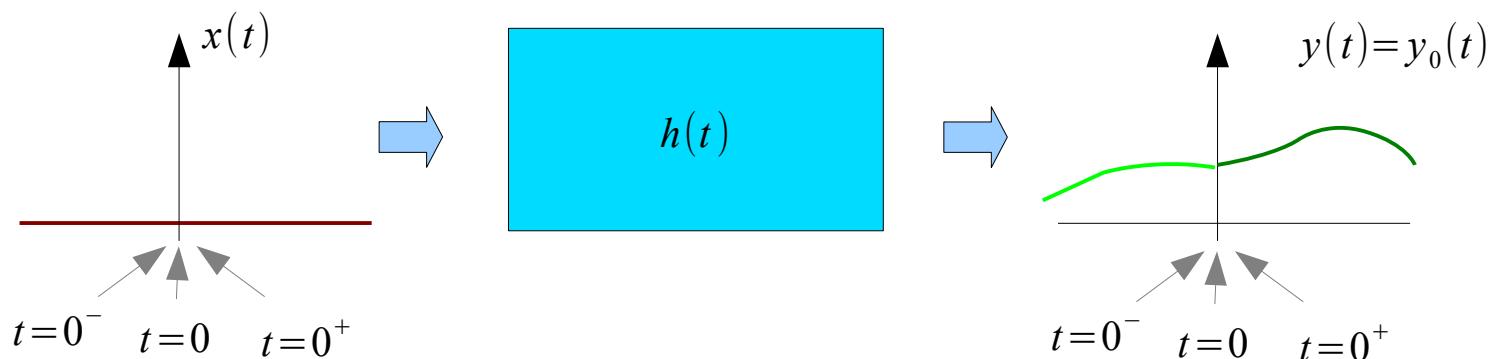
$$y(t) = \underbrace{y_n(t)}_{\text{Natural Response}} + \underbrace{y_\Phi(t)}_{\text{Forced Response}}$$

Zero Input Response

$$\frac{d^N y(t)}{d t^N} + \color{red}{a_1} \frac{d^{N-1} y(t)}{d t^{N-1}} + \cdots + \color{red}{a_{N-1}} \frac{d y(t)}{d t} + \color{red}{a_N} y(t) = \color{green}{b_{N-M}} \frac{d^M x(t)}{d t^M} + \color{green}{b_{N-M+1}} \frac{d^M x(t)}{d t^{M-1}} + \cdots + \color{green}{b_{N-1}} \frac{d x(t)}{d t} + \color{green}{b_N} x(t)$$

$$(\mathcal{D}^N + \color{red}{a_1} \mathcal{D}^{N-1} + \cdots + \color{red}{a_{N-1}} \mathcal{D} + \color{red}{a_N}) y(t) = (\mathcal{D}^M + \color{green}{b_{N-M+1}} \mathcal{D}^{M-1} + \cdots + \color{green}{b_{N-1}} \mathcal{D} + \color{green}{b_N}) x(t)$$

$$Q(\mathcal{D}) y(t) = P(\mathcal{D}) x(t)$$



Input is zero

$$y_0(0^-) = y_0(0) = y_0(0^+)$$

Only initial conditions
drives the system

$$\dot{y}_0(0^-) = \dot{y}_0(0) = \dot{y}_0(0^+)$$

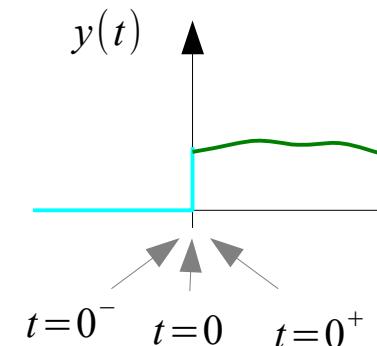
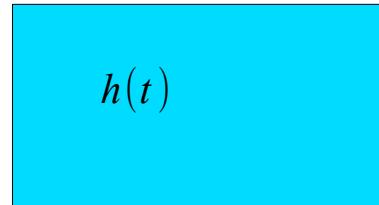
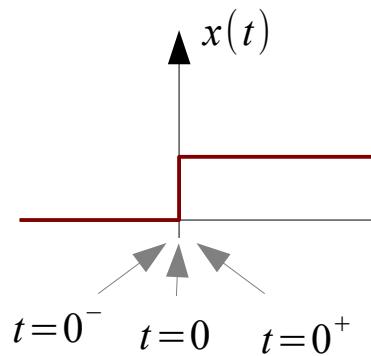
$$\ddot{y}_0(0^-) = \ddot{y}_0(0) = \ddot{y}_0(0^+)$$

Zero State Response

$$\frac{d^N y(t)}{d t^N} + \color{red}{a_1} \frac{d^{N-1} y(t)}{d t^{N-1}} + \cdots + \color{red}{a_{N-1}} \frac{d y(t)}{d t} + \color{red}{a_N} y(t) = \color{green}{b_{N-M}} \frac{d^M x(t)}{d t^M} + \color{green}{b_{N-M+1}} \frac{d^M x(t)}{d t^{M-1}} + \cdots + \color{green}{b_{N-1}} \frac{d x(t)}{d t} + \color{green}{b_N} x(t)$$

$$(\mathcal{D}^N + \color{red}{a_1} \mathcal{D}^{N-1} + \cdots + \color{red}{a_{N-1}} \mathcal{D} + \color{red}{a_N}) y(t) = (\mathcal{D}^M + \color{green}{b_{N-M+1}} \mathcal{D}^{M-1} + \cdots + \color{green}{b_{N-1}} \mathcal{D} + \color{green}{b_N}) x(t)$$

$$Q(\mathcal{D}) y(t) = P(\mathcal{D}) x(t)$$



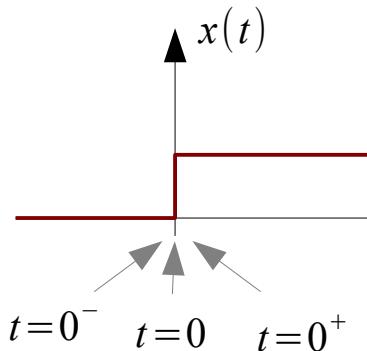
All initial conditions are zero

Total Response $y(t)$

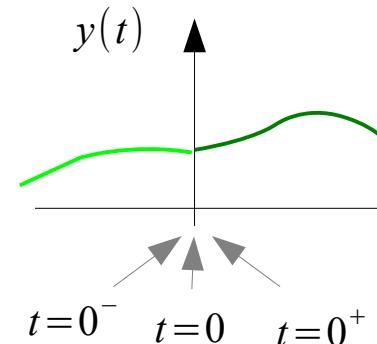
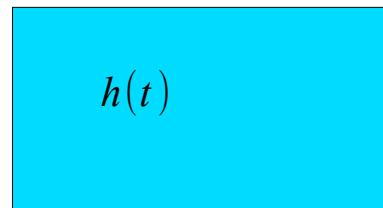
$$\frac{d^N y(t)}{d t^N} + \color{red}{a_1} \frac{d^{N-1} y(t)}{d t^{N-1}} + \cdots + \color{red}{a_{N-1}} \frac{d y(t)}{d t} + \color{red}{a_N} y(t) = \color{green}{b_{N-M}} \frac{d^M x(t)}{d t^M} + \color{green}{b_{N-M+1}} \frac{d^M x(t)}{d t^{M-1}} + \cdots + \color{green}{b_{N-1}} \frac{d x(t)}{d t} + \color{green}{b_N} x(t)$$

$$(\mathcal{D}^N + \color{red}{a_1} \mathcal{D}^{N-1} + \cdots + \color{red}{a_{N-1}} \mathcal{D} + \color{red}{a_N}) y(t) = (\mathcal{D}^M + \color{green}{b_{N-M+1}} \mathcal{D}^{M-1} + \cdots + \color{green}{b_{N-1}} \mathcal{D} + \color{green}{b_N}) x(t)$$

$$Q(\mathcal{D}) y(t) = P(\mathcal{D}) x(t)$$



zero input response
+
zero state response



$$y(t) = y_0(t) \quad t \leq 0^-$$

because the input
has not started yet

$$y(0^-) = y_0(0^-)$$

$$\dot{y}(0^-) = \dot{y}_0(0^-)$$

The total response

$$y(0^-) = y(0^+)$$

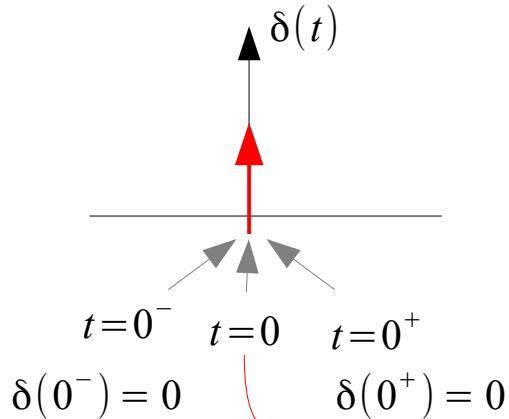
$$\dot{y}(0^-) = \dot{y}(0^+)$$

Impulse Response $h(t)$

$$\frac{d^N y(t)}{d t^N} + \color{red}{a_1} \frac{d^{N-1} y(t)}{d t^{N-1}} + \cdots + \color{red}{a_{N-1}} \frac{d y(t)}{d t} + \color{red}{a_N} y(t) = \color{green}{b_{N-M}} \frac{d^M x(t)}{d t^M} + \color{green}{b_{N-M+1}} \frac{d^M x(t)}{d t^{M-1}} + \cdots + \color{green}{b_{N-1}} \frac{d x(t)}{d t} + \color{green}{b_N} x(t)$$

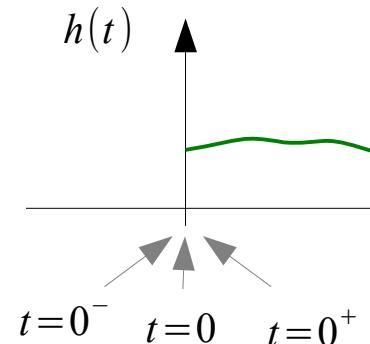
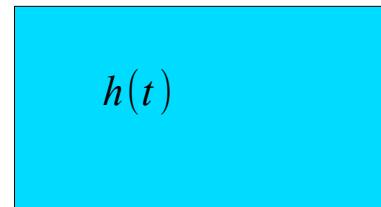
$$(\mathcal{D}^N + \color{red}{a_1} \mathcal{D}^{N-1} + \cdots + \color{red}{a_{N-1}} \mathcal{D} + \color{red}{a_N}) y(t) = (\mathcal{D}^M + \color{green}{b_{N-M+1}} \mathcal{D}^{M-1} + \cdots + \color{green}{b_{N-1}} \mathcal{D} + \color{green}{b_N}) x(t)$$

$$Q(\mathcal{D}) y(t) = P(\mathcal{D}) x(t)$$



All init conditions
are zero at $t=0^-$

Generates energy storage
Creates nonzero initial
condition at $t=0^+$



$h(t) = \text{characteristic mode terms}$
 $t \geq 0^+ \quad (t \neq 0)$

At $t=0$, at most impulse $A_0 \delta(t)$

Impulse Response $h(t)$

$$\frac{d^N y(t)}{dt^N} + \color{red}{a_1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + \color{red}{a_{N-1}} \frac{d y(t)}{dt} + \color{red}{a_N} y(t) = \color{green}{b_0} \frac{d^M x(t)}{dt^M} + \color{green}{b_1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + \color{green}{b_{N-1}} \frac{d x(t)}{dt} + \color{green}{b_N} x(t)$$

$$(\color{blue}{D}^N + \color{red}{a_1} \color{blue}{D}^{N-1} + \cdots + \color{red}{a_{N-1}} \color{blue}{D} + \color{red}{a_N}) y(t) = (\color{green}{b_0} \color{blue}{D}^M + \color{green}{b_1} \color{blue}{D}^{M-1} + \cdots + \color{green}{b_{N-1}} \color{blue}{D} + \color{green}{b_N}) x(t)$$

$$M = N$$

$$\mathcal{Q}(\color{blue}{D}) y(t) = P(\color{blue}{D}) x(t)$$

If $\delta(t)$ is included in $h(t)$

$$\underbrace{(\color{blue}{D}^N + \color{red}{a_1} \color{blue}{D}^{N-1} + \cdots + \color{red}{a_{N-1}} \color{blue}{D} + \color{red}{a_N}) h(t)}_{\downarrow} = \underbrace{(\color{green}{b_0} \color{blue}{D}^M + \color{green}{b_1} \color{blue}{D}^{M-1} + \cdots + \color{green}{b_{N-1}} \color{blue}{D} + \color{green}{b_N}) \delta(t)}_{\downarrow}$$

The highest order term $\delta^{(N+1)}(t)$ \longleftrightarrow $\delta^{(N)}(t)$ contradiction

$h(t)$ cannot contain $\delta^{(i)}(t)$ at all
 $h(t)$ can contain at most $\delta(t)$

Simplified Impulse Matching Method (1)

$$\frac{d^N y(t)}{dt^N} + \color{red}{a_1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + \color{red}{a_{N-1}} \frac{d y(t)}{dt} + \color{red}{a_N} y(t) = \color{green}{b_0} \frac{d^M x(t)}{dt^M} + \color{green}{b_1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + \color{green}{b_{N-1}} \frac{d x(t)}{dt} + \color{green}{b_N} x(t)$$

$$(\color{blue}{D}^N + \color{red}{a_1} \color{blue}{D}^{N-1} + \cdots + \color{red}{a_{N-1}} \color{blue}{D} + \color{red}{a_N}) y(t) = (\color{green}{b_0} \color{blue}{D}^M + \color{green}{b_1} \color{blue}{D}^{M-1} + \cdots + \color{green}{b_{N-1}} \color{blue}{D} + \color{green}{b_N}) x(t)$$

$$\color{teal}{M} = N$$

$$Q(\color{blue}{D}) y(t) = P(\color{blue}{D}) x(t)$$

$$h(t) = b_0 \delta(t) + [P(D) y_n(t)] u(t)$$

$y_n(t)$ Linear combination of characteristic modes with the following initial conditions

$$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) \cdots = y_n^{(N-2)}(0) = 0 \quad y_n^{(N-1)}(0) = 1$$

$$(\color{blue}{D}^N + \color{red}{a_1} \color{blue}{D}^{N-1} + \cdots + \color{red}{a_{N-1}} \color{blue}{D} + \color{red}{a_N}) y_n(t) = \delta(t)$$

$$y_n^{(N)}(t) + \color{red}{a_1} y_n^{(N-1)}(t) + \cdots + \color{red}{a_{N-1}} y_n^{(1)}(t) + y_n(t) = \delta(t)$$

$$Q(\color{blue}{D}) y(t) = P(\color{blue}{D}) x(t)$$

$$P(\color{blue}{D}) \leftarrow 1$$

$$Q(\color{blue}{D}) w(t) = x(t)$$

$$Q(\color{blue}{D}) y_n(t) = \delta(t)$$

$$Q(\color{blue}{D}) w(t) = x(t)$$

$$Q(\color{blue}{D}) P(\color{blue}{D}) w(t) = P(\color{blue}{D}) x(t)$$

$$Q(\color{blue}{D}) y(t) = P(\color{blue}{D}) x(t)$$

Simplified Impulse Matching Method (2)

$$\frac{d^N y(t)}{d t^N} + \color{red}{a_1} \frac{d^{N-1} y(t)}{d t^{N-1}} + \cdots + \color{red}{a_{N-1}} \frac{d y(t)}{d t} + \color{red}{a_N} y(t) = \color{green}{b_0} \frac{d^M x(t)}{d t^M} + \color{green}{b_1} \frac{d^M x(t)}{d t^{M-1}} + \cdots + \color{green}{b_{N-1}} \frac{d x(t)}{d t} + \color{green}{b_N} x(t)$$

$$(\color{blue}{D}^N + \color{red}{a_1} \color{blue}{D}^{N-1} + \cdots + \color{red}{a_{N-1}} \color{blue}{D} + \color{red}{a_N}) y(t) = (\color{green}{b_0} \color{blue}{D}^M + \color{green}{b_1} \color{blue}{D}^{M-1} + \cdots + \color{green}{b_{N-1}} \color{blue}{D} + \color{green}{b_N}) x(t)$$

$$\color{teal}{M} = N$$

$$Q(\color{blue}{D}) y(t) = P(\color{blue}{D}) x(t)$$

$$(\color{blue}{D}^N + \color{red}{a_1} \color{blue}{D}^{N-1} + \cdots + \color{red}{a_{N-1}} \color{blue}{D} + \color{red}{a_N}) y_n(t) = \delta(t)$$

$$y_n^{(N)}(t) + \color{red}{a_1} y_n^{(N-1)}(t) + \cdots + \color{red}{a_{N-1}} y_n^{(1)}(t) + y_n(t) = \delta(t)$$

$$Q(\color{blue}{D}) w(t) = x(t)$$

$$Q(\color{blue}{D}) P(\color{blue}{D}) w(t) = P(\color{blue}{D}) x(t)$$

$$Q(\color{blue}{D}) y(t) = P(\color{blue}{D}) x(t)$$

$$h(t) = P(D)[y_n(t)u(t)]$$

$$h(t) = b_o \delta(t) + P(D)y_n(t), \quad t \geq 0$$

$$h(t) = b_o \delta(t) + [P(D)y_n(t)]u(t)$$

Impulse Response $h(t)$

References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] B.P. Lathi, Linear Systems and Signals (2nd Ed)