DTFT (3A)

- Discrete Time Fourier Transform

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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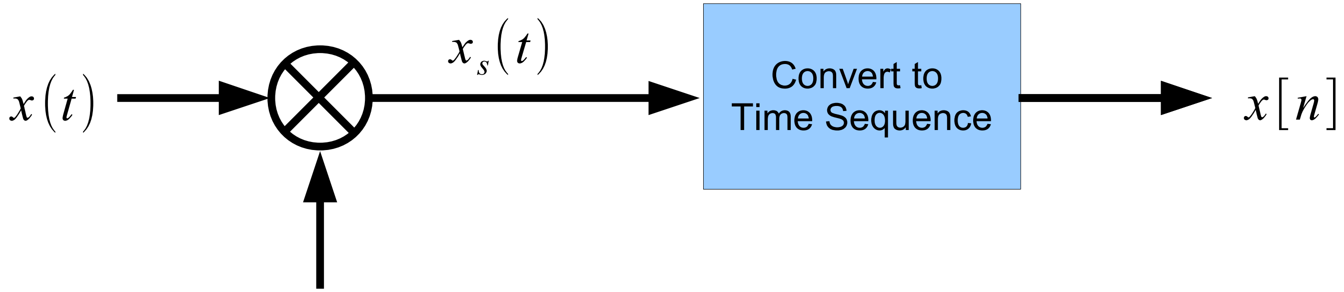
Sampling and Reconstruction

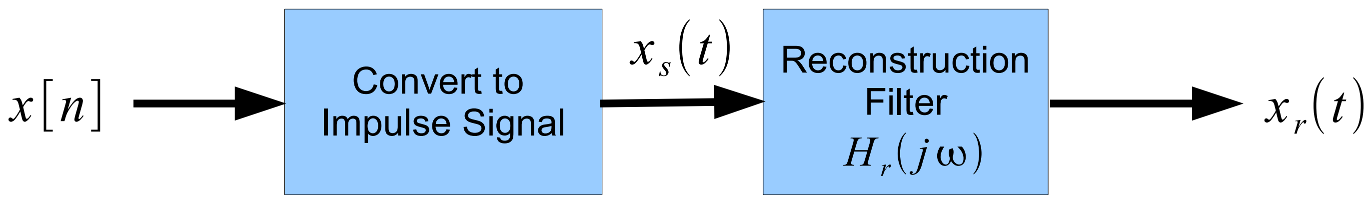
Ideal Sampling



$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

T_s **Sampling Period**

Ideal Reconstruction



CTFS of Impulse Train (1)

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

Fourier Series →

$$p(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{+jk\omega_s t}$$

Fourier Series Expansion

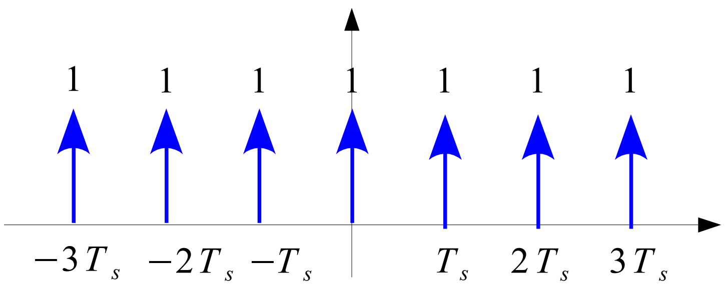
$$p(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t}$$

Fourier Series Coefficients

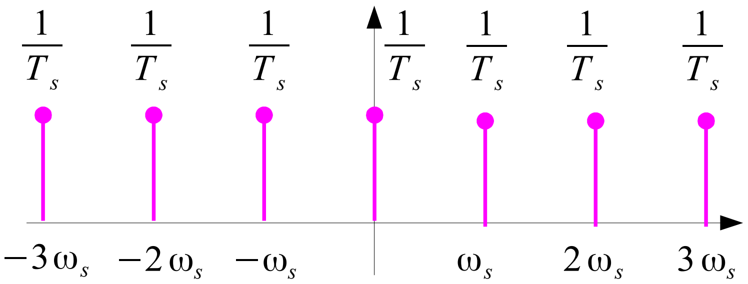
$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t)\, e^{-jk\omega_s t}\, dt$$

$$= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t)\, e^{-jk\omega_s 0}\, dt$$

$$= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t)\, dt = \frac{1}{T_s}$$



$$\omega_s = \frac{2\pi}{T_s}$$



CTFS of Impulse Train (2)

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

Fourier Series

$$p(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{+jk\omega_s t}$$

$$p(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t}$$

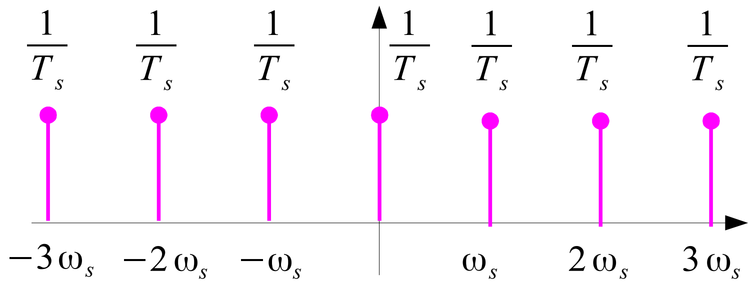
Fourier Transform

$$P(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \, \delta(\omega - k\omega_s)$$

Fourier Transform of impulse train

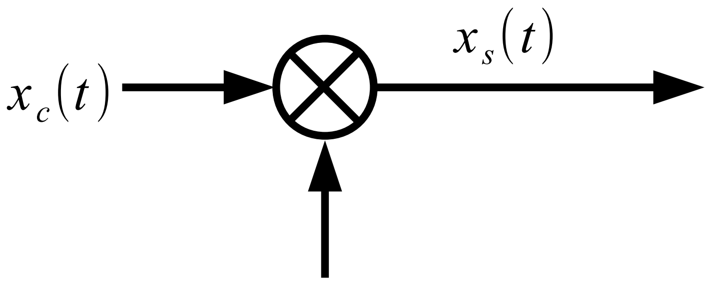
$$P(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T_s} \delta(\omega - k\omega_s)$$

$$\omega_s = \frac{2\pi}{T_s}$$



Sampled Signal

Ideal Sampling



$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{+\infty} x_c(nT_s)\delta(t - nT_s)$$

$\Leftrightarrow$

$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT Frequency Shift Property

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega \quad \Longleftrightarrow \quad X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

Frequency Shift Property

$$x_c(t) \quad \Longleftrightarrow \quad X_c(j\omega)$$

$$x_c(t)\, e^{jk\omega_s t} \quad \Longleftrightarrow \quad X_c(j(\omega - k\omega_s))$$

$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t} \quad \Longleftrightarrow \quad X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s} \qquad\qquad \omega_s = \frac{2\pi}{T_s}$$

CTFT Delay Property

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega \quad \Longleftrightarrow \quad X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

Fourier Transform of an Impulse

$$\delta(t - t_d) \quad \Longleftrightarrow \quad e^{j\omega t_d}$$

$$\delta(t - nT_s) \quad \Longleftrightarrow \quad e^{-j\omega n T_s}$$

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\,\delta(t - nT_s) \quad \Longleftrightarrow \quad X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

CTFT of a Sampled Signal

Continuous Time Fourier <u>**Transform**</u>

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega \quad \Leftrightarrow \quad X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\,\delta(t - nT_s)$$

CTFT $\Leftrightarrow$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

CTFS ||

DTFT ||

$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT $\Leftrightarrow$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

z-Transform of a Sampled Signal

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

$=$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} x_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT of a sampled signal

$$\sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s} = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s)) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - \frac{2\pi k}{T_s}))$$

Z-Transform of a sampled signal

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad x[n] = x_c(nT_s)$$

$$X(z)\Big|_{z=e^{j\omega T_s}} = X(e^{j\omega T_s})$$

evaluated at $z = e^{j\omega T_s}$

z-Transform and Normalized Frequency

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

=

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} x_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X(z)\Big|_{z=e^{j\omega T_s}} = X(e^{j\omega T_s}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

z-Transform

$$\hat{\omega} = \omega T_s$$

Normalized Frequency

$$X(z)\Big|_{z=e^{j\hat{\omega}}} \quad X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

Discrete Time Fourier Transform

DTFT and CTFT

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

$\Longleftrightarrow$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} x_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

DTFT of a sampled signal

$$X(e^{j\hat{\omega}}) \Big|_{\hat{\omega} = \omega T_s} = X(e^{j\omega T_s}) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$= \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c\left(j\left(\omega - \frac{2\pi k}{T_s}\right)\right)$$

CTFT of a sampled signal

DTFT and CTFT

Continuous Time Fourier Transform CTFT

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Discrete Time Fourier Transform DTFT

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \quad \Longleftrightarrow \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\hat{\omega}})\, e^{+j\hat{\omega} n}\, d\hat{\omega}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$= \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

=

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

Dirichlet Function (1)

Dirichlet Function

$$drcl(t, L) = \frac{\sin(\pi L t)}{L \sin(\pi t)}$$

$$diric(x, N) = \frac{\sin(N x/2)}{N \sin(x/2)}$$

$$D_L(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega} L/2)}{L \sin(\hat{\omega}/2)}$$

odd L ➡ Infinite sum of uniformly spaced sinc functions

$t = \frac{m}{L}$ ➡ $\sin(\pi L t) = 0$ integer multiples of L

$t = n$ ➡ $\sin(\pi t) = 0$ integer n

$$\lim_{t \to n} \frac{\sin(\pi L t)}{L \sin(\pi t)} = \lim_{t \to n} \frac{L \pi \cos(\pi L t)}{L \pi \cos(\pi t)} = \pm 1$$

$n = -2$	$n = -1$	$n = 0$	$n = +1$	$n = +2$	$n = +3$	
$\frac{\cos(-L 2\pi)}{\cos(-2\pi)}$	$\frac{\cos(-L\pi)}{\cos(-\pi)}$	$\frac{\cos(0)}{\cos(0)}$	$\frac{\cos(L\pi)}{\cos(\pi)}$	$\frac{\cos(L 2\pi)}{\cos(2\pi)}$	$\frac{\cos(L 3\pi)}{\cos(3\pi)}$	$\cdots$
$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	odd L
$+1$	-1	$+1$	-1	$+1$	-1	even L

odd L $\lim_{t \to n} \frac{\cos(\pi L t)}{\cos(\pi t)} = +1$

even L $\lim_{t \to n} \frac{\cos(\pi L t)}{\cos(\pi t)} = (-1)^n$

Dirichlet Function (2)

Dirichlet Function

$$drcl(t, L) = \frac{\sin(\pi L t)}{L \sin(\pi t)}$$

$$diric(x, N) = \frac{\sin(N x/2)}{N \sin(x/2)}$$

$$D_L(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega} L/2)}{L \sin(\hat{\omega}/2)}$$

$$D_L(e^{j(\hat{\omega}+2\pi)}) = \frac{\sin((\hat{\omega}+2\pi)L/2)}{L\sin((\hat{\omega}+2\pi)/2)} = \frac{\sin(\hat{\omega}L/2 + L\pi)}{L\sin(\hat{\omega}/2+\pi)}$$

$$\begin{cases} +D_L(e^{j\hat{\omega}}) & \textit{for an odd } L \\ -D_L(e^{j\hat{\omega}}) & \textit{for an even } L \end{cases}$$

(period: 2π)

$0 \le \hat{\omega} \le +\pi$

$0 \le \hat{\omega}/2 \le +\frac{\pi}{2}$ $\qquad$ $0 \le \hat{\omega} L/2 \le +L\frac{\pi}{2}$

$0 \le \sin(\hat{\omega}/2) \le +1$ $\qquad$ $-1 \le \sin(\hat{\omega} L/2) \le +1$

a quarter period $\qquad$ L quarter periods

envelope: $\dfrac{1}{\sin(\hat{\omega}/2)}$ $\qquad$ zeros: $\hat{\omega} = \dfrac{2\pi}{L}k$

$\sin(\hat{\omega}L/2) = 0$

$$D_L(e^{-j\hat{\omega}}) = \frac{\sin(-\hat{\omega}L/2)}{L\sin(-\hat{\omega}/2)} = D_L(e^{j\hat{\omega}})$$

an even function

$$\lim_{\hat{\omega}\to 0} D_L(e^{j\hat{\omega}}) = \lim_{\hat{\omega}\to 0} \frac{L/2\cos(\hat{\omega}L/2)}{L/2\cos(\hat{\omega}/2)} = 1$$

max value $D_L(e^{j\hat{\omega}}) = 1$ *when* $\hat{\omega} = 0$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003