

DTFT (3A)

- Discrete Time Fourier Transform

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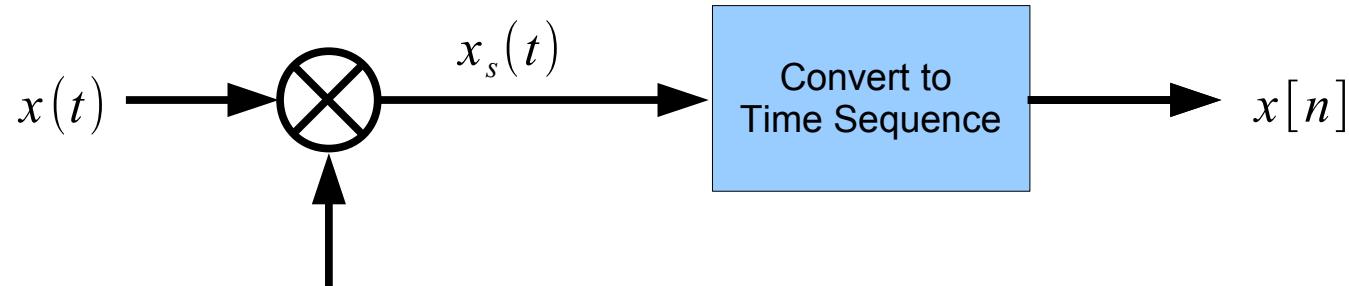
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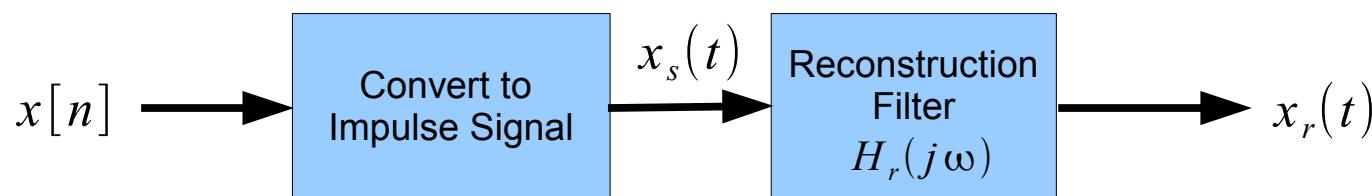
Sampling and Reconstruction

Ideal Sampling



$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \quad T_s \text{ Sampling Period}$$

Ideal Reconstruction



CTFS of Impulse Train (1)

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

Fourier Series 

$$p(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{+jk\omega_s t}$$

Fourier Series Expansion

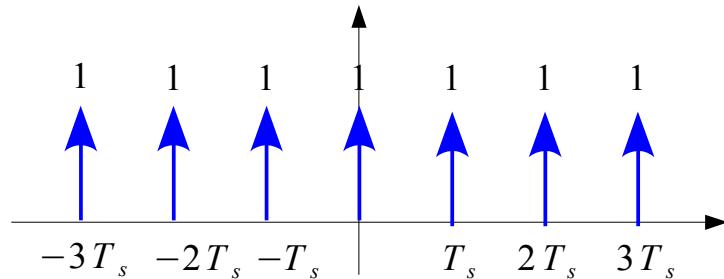
$$p(t) = \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_s t}$$

Fourier Series Coefficients

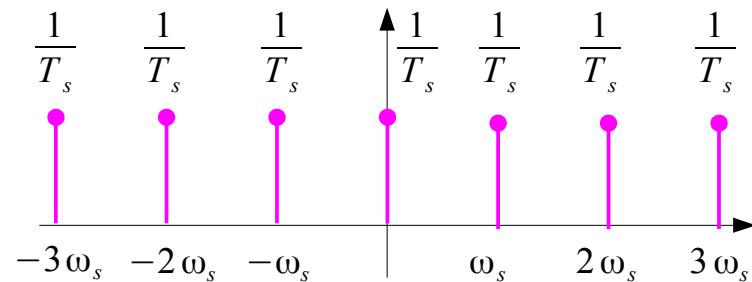
$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) e^{-jk\omega_s t} dt$$

$$= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) e^{-jk\omega_s 0} dt$$

$$= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) dt = \frac{1}{T_s}$$



$$\omega_s = \frac{2\pi}{T_s}$$



CTFS of Impulse Train (2)

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

Fourier Series

$$p(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{+jk\omega_s t}$$

$$p(t) = \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_s t}$$

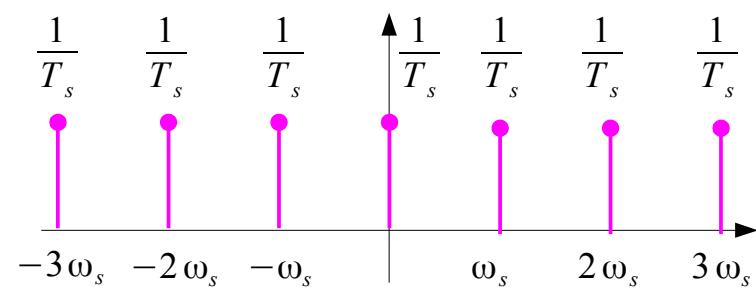
Fourier Transform

$$P(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_s)$$

Fourier Transform of impulse train

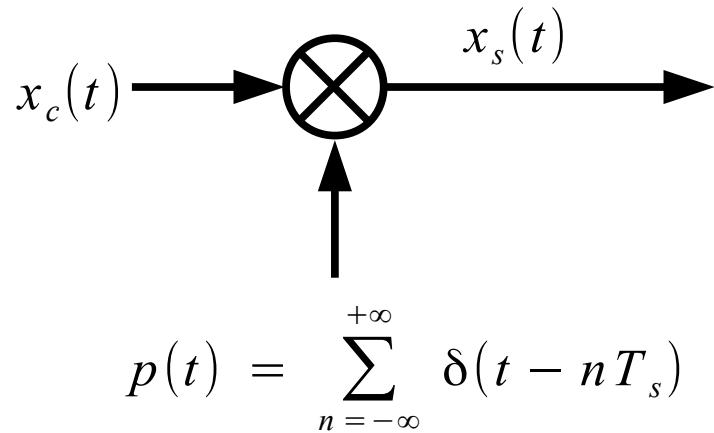
$$P(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T_s} \delta(\omega - k\omega_s)$$

$$\omega_s = \frac{2\pi}{T_s}$$



Sampled Signal

Ideal Sampling



$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{+\infty} x_c(nT_s) \delta(t - nT_s)$$

$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT Frequency Shift Property

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

Frequency Shift Property

$$x_c(t)$$



$$X_c(j\omega)$$

$$x_c(t) e^{jk\omega_s t}$$



$$X_c(j(\omega - k\omega_s))$$

$$x_s(t) = \textcolor{blue}{x}_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} \textcolor{blue}{X}_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT Delay Property

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

Fourier Transform of an Impulse

$$\delta(t - t_d)$$



$$e^{j\omega t_d}$$

$$\delta(t - nT_s)$$



$$e^{-j\omega nT_s}$$

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$



$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) e^{-j\omega nT_s}$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) \delta(t - nT_s)$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega nT_s}$$

CTFT of a Sampled Signal

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

CTFT



$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) \delta(t - nT_s)$$

CTFS



$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$

CTFT



$$\omega_s = \frac{2\pi}{T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) e^{-jn\omega T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jn\omega T_s}$$

DTFT



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

z-Transform of a Sampled Signal

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) e^{-j\omega nT_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega nT_s}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} x_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT of a sampled signal

$$\sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega nT_s} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s)) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - \frac{2\pi k}{T_s}))$$

Z-Transform of a sampled signal



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad x[n] = x_c(nT_s)$$

$$X(z) \Big|_{z = e^{j\omega T_s}} = X(e^{j\omega T_s})$$

evaluated at $z = e^{j\omega T_s}$

z-Transform and Normalized Frequency

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) e^{-j\omega nT_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega nT_s}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} x_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X(z) \Big|_{z = e^{j\omega T_s}} = X(e^{j\omega T_s}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega nT_s}$$

z-Transform



$$\hat{\omega} = \omega T_s$$

**Normalized
Frequency**

$$X(z) \Big|_{z = e^{j\hat{\omega}}} = X(e^{j\hat{\omega}})$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}n}$$

**Discrete Time
Fourier Transform**

DTFT and CTFT

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s) e^{-j\omega nT_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega nT_s}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} x_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}n}$$

DTFT of a sampled signal

$$X(e^{j\hat{\omega}}) \Big|_{\hat{\omega} = \omega T_s} = X(e^{j\omega T_s}) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - \frac{2\pi k}{T_s}))$$

CTFT of a sampled signal

DTFT and CTFT

Continuous Time Fourier Transform

CTFT

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \leftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

Discrete Time Fourier Transform

DTFT

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}n} \quad \leftrightarrow \quad x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\hat{\omega}}) e^{+j\hat{\omega}n} d\hat{\omega}$$

$$\begin{aligned} X_s(j\omega) &= \sum_{n=-\infty}^{+\infty} x_c(nT_s) e^{-j\omega nT_s} \\ &= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega nT_s} \end{aligned}$$



$$\begin{aligned} X_s(j\omega) &= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s)) \\ \omega_s &= \frac{2\pi}{T_s} \end{aligned}$$

Dirichlet Function (1)

Dirichlet Function

$$drcl(t, L) = \frac{\sin(\pi L t)}{L \sin(\pi t)}$$

$$diric(x, N) = \frac{\sin(Nx/2)}{N \sin(x/2)}$$

$$D_L(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}L/2)}{L \sin(\hat{\omega}/2)}$$

odd L \rightarrow Infinite sum of uniformly spaced sinc functions

$$t = \frac{m}{L} \quad \rightarrow \quad \sin(\pi L t) = 0 \quad \text{integer multiples of } L$$

$$t = n \quad \rightarrow \quad \sin(\pi t) = 0 \quad \text{integer } n \quad \lim_{t \rightarrow n} \frac{\sin(\pi L t)}{L \sin(\pi t)} = \lim_{t \rightarrow n} \frac{L \pi \cos(\pi L t)}{L \pi \cos(\pi t)} = \pm 1$$

$$\begin{array}{ccccccc} n = -2 & & n = -1 & & n = 0 & & n = +1 & & n = +2 & & n = +3 \\ \downarrow & & \downarrow \end{array}$$

$$\frac{\cos(-L2\pi)}{\cos(-2\pi)}, \frac{\cos(-L\pi)}{\cos(-\pi)}, \frac{\cos(0)}{\cos(0)}, \frac{\cos(L\pi)}{\cos(\pi)}, \frac{\cos(L2\pi)}{\cos(2\pi)}, \frac{\cos(L3\pi)}{\cos(3\pi)}, \dots$$

$$\begin{array}{ccccccc} \text{odd L} & \lim_{t \rightarrow n} \frac{\cos(\pi L t)}{\cos(\pi t)} = +1 & & +1 & & +1 & & +1 & & +1 & & \text{odd L} \\ & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \end{array}$$

$$\begin{array}{ccccccc} \text{even L} & \lim_{t \rightarrow n} \frac{\cos(\pi L t)}{\cos(\pi t)} = (-1)^n & & +1 & & -1 & & +1 & & -1 & & \text{even L} \\ & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \end{array}$$

Dirichlet Function (2)

Dirichlet Function

$$drcl(t, L) = \frac{\sin(\pi L t)}{L \sin(\pi t)}$$

$$diric(x, N) = \frac{\sin(\textcolor{red}{N} x/2)}{\textcolor{red}{N} \sin(x/2)}$$

$$D_{\textcolor{red}{L}}(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega} L/2)}{L \sin(\hat{\omega}/2)}$$

$$\begin{aligned} D_{\textcolor{red}{L}}(e^{j(\hat{\omega} + 2\pi)}) &= \frac{\sin((\hat{\omega} + 2\pi)L/2)}{L \sin((\hat{\omega} + 2\pi)/2)} \\ &= \frac{\sin(\hat{\omega} L/2 + L\pi)}{L \sin(\hat{\omega}/2 + \pi)} \end{aligned}$$

$$\begin{cases} +D_{\textcolor{red}{L}}(e^{j\hat{\omega}}) & \text{for an odd } L \quad (\text{period: } 2\pi) \\ -D_{\textcolor{red}{L}}(e^{j\hat{\omega}}) & \text{for an even } L \end{cases}$$

$$0 \leq \hat{\omega} \leq +\pi$$

$$\begin{aligned} 0 \leq \hat{\omega}/2 &\leq +\frac{\pi}{2} & 0 \leq \hat{\omega} L/2 &\leq +\frac{L\pi}{2} \\ 0 \leq \sin(\hat{\omega}/2) &\leq +1 & -1 \leq \sin(\hat{\omega} L/2) &\leq +1 \end{aligned}$$

a quarter period

$$\begin{aligned} \text{envelope: } \frac{1}{\sin(\hat{\omega}/2)} && \text{zeros: } \hat{\omega} = \frac{2\pi}{L} k \\ \sin(\hat{\omega} L/2) &= 0 \end{aligned}$$

$$D_{\textcolor{red}{L}}(e^{-j\hat{\omega}}) = \frac{\sin(-\hat{\omega} L/2)}{L \sin(-\hat{\omega}/2)} = D_{\textcolor{red}{L}}(e^{j\hat{\omega}})$$

an even function

$$\lim_{\hat{\omega} \rightarrow 0} D_{\textcolor{red}{L}}(e^{j\hat{\omega}}) = \lim_{\hat{\omega} \rightarrow 0} \frac{L/2 \cos(\hat{\omega} L/2)}{L/2 \cos(\hat{\omega}/2)} = 1$$

max value $D_{\textcolor{red}{L}}(e^{j\hat{\omega}}) = 1$ when $\hat{\omega} = 0$

References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., *Signal Processing First*, Pearson Prentice Hall, 2003