General Vector Space (3A)

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Vector Space

V: non-empty set of objects

defined operations: addition $\mathbf{u} + \mathbf{v}$

scalar multiplication $k\,\mathbf{u}$

if the following axioms are satisfied

for all object $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and all scalar k, m ⇨ V: vector space

objects in V: vectors

1. if $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u} + \mathbf{v}$ is in V
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ (zero vector)
5. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + (\mathbf{u}) = \mathbf{0}$
6. if k is any scalar and $\mathbf{u}$ is objects in V, then $k\mathbf{u}$ is in V
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)\mathbf{u}$
10. $1(\mathbf{u}) = \mathbf{u}$

Test for a Vector Space

1. Identify the set V of objects
2. Identify the addition and scalar multiplication on V
3. Verify $\mathbf{u} + \mathbf{v}$ is in V and $k\mathbf{u}$ is in V
   closure under **addition** and **scalar multiplication**
4. Confirm other axioms.

1. if $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u} + \mathbf{v}$ is in V
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ (zero vector)
5. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + (\mathbf{u}) = \mathbf{0}$
6. if k is any scalar and $\mathbf{u}$ is objects in V, then $k\mathbf{u}$ is in V
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)\mathbf{u}$
10. $1(\mathbf{u}) = \mathbf{u}$

Subspace

a subset W of a vector space V

If the subset W is itself a vector space ⇒ the subset W is a **subspace** of V

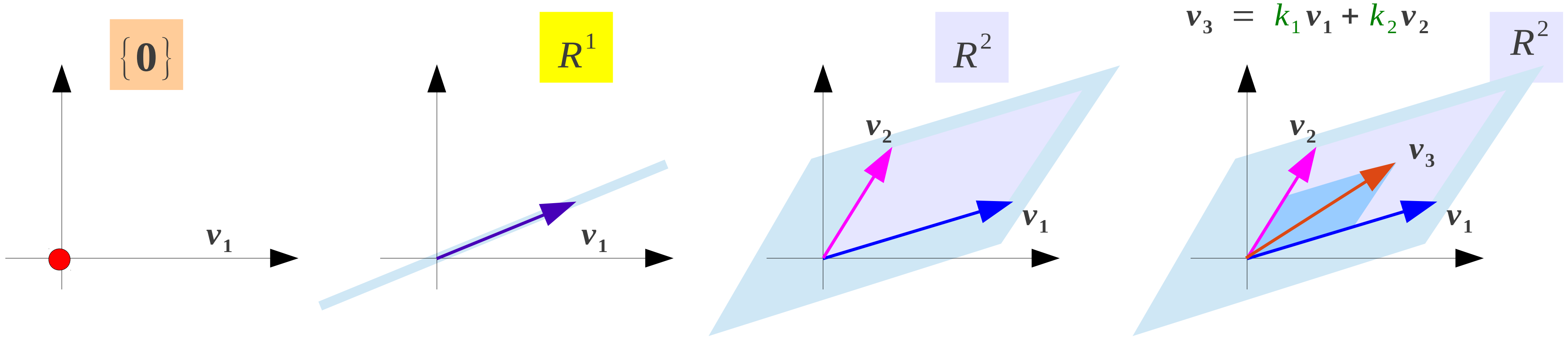
1. if **u** and **v** are objects in W, then **u** + **v** is in W
2. **u** + **v** = **v** + **u**
3. **u** + (**v** + **w**) = (**u** + **v**) + **w**
4. **0** + **u** = **u** + **0** = **u** (zero vector)
5. **u** + (–**u**) = (–**u**) + (**u**) = **0**
6. if *k* is any scalar and **u** is objects in W, then *k***u** is in W
7. *k*(**u** + **v**) = *k***u** + *k***v**
8. (*k* + *m*)**u** = *k***u** + *m***u**
9. *k*(*m***u**) = (*km*)**u**
10. *1*(**u**) = **u**

Subspace Example (1)

In vector space R^2

any one vector	(linearly indep.)	**spans**	R^1	line through **0**
any two non-collinear vectors	(linearly indep.)	**spans**	R^2	plane
any three or more vectors	(linearly dep.)	**spans**	R^2	plane

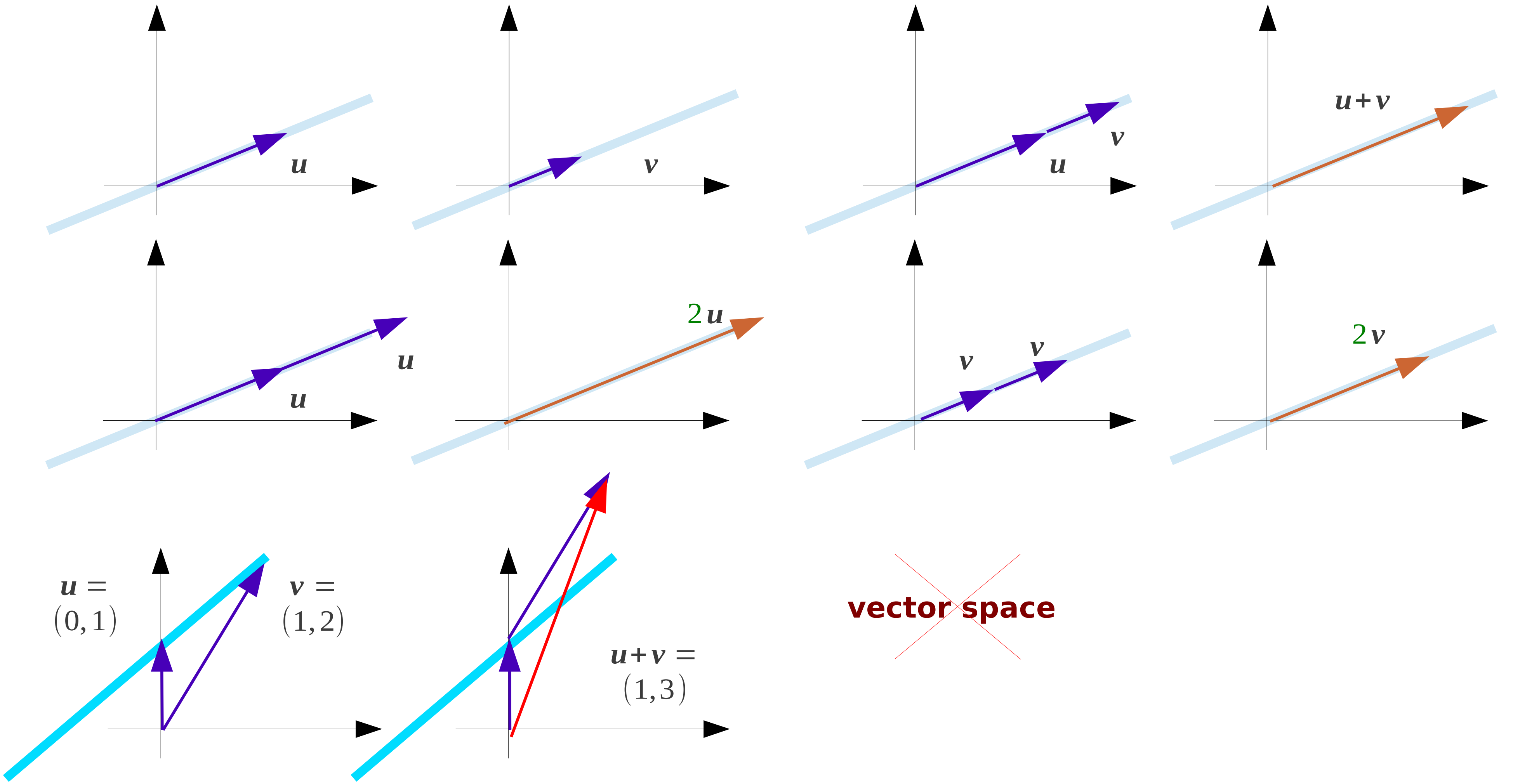
Subspaces of R^2



Subspace Example (2)

In vector space R^2

any one vector (linearly indep.) **spans** R^1 line through **0**



~~**vector space**~~

Subspace Example (3)

In vector space R^3

any one vector	(linearly indep.)	**spans**	R^1	line through **0**
any two non-collinear vectors	(linearly indep.)	**spans**	R^2	plane through **0**
any three vectors non-collinear, non-coplanar	(linearly indep.)	**spans**	R^3	3-dim space
any four or more vectors	(linearly dep.)	**spans**	R^3	3-dim space

Subspaces of R^3

$\{\mathbf{0}\}$	R^1	R^2	R^3
	line through **0**	plane through **0**	3-dim space

Row & Column Spaces

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

ROW Space **subspace** of R^n

$$= span\{\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_m\}$$

COLUMN Space **subspace** of R^m

$$= span\{\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_n\}$$

$$\begin{aligned} \boldsymbol{r}_1 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \\ \boldsymbol{r}_2 &= \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \\ & \quad\ \ \vdots \qquad \vdots \qquad\qquad\quad \vdots \\ \boldsymbol{r}_m &= \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{aligned}$$

$$\overset{\longleftrightarrow}{\boldsymbol{n}}$$

$$\boldsymbol{r}_i \in R^n$$

$$\begin{array}{cccc} \boldsymbol{c}_1 & \boldsymbol{c}_2 & & \boldsymbol{c}_n \\ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} & \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \end{array} \updownarrow \boldsymbol{m}$$

$$\boldsymbol{c}_i \in R^m$$

Row Space

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

ROW Space **subspace** of R^n

$$= span\{\boldsymbol{r_1}, \boldsymbol{r_2}, \cdots, \boldsymbol{r_m}\}$$

$$= \{\boldsymbol{w}\}$$

$$\boldsymbol{r_i} \in R^n$$

$$\boldsymbol{w} = k_1\boldsymbol{r_1} + k_2\boldsymbol{r_2} + \cdots + k_m\boldsymbol{r_m}$$

$$\begin{aligned} \boldsymbol{r_1} &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \\ \boldsymbol{r_2} &= \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \\ &\ \ \vdots \\ \boldsymbol{r_m} &= \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{aligned}$$

$$\longleftrightarrow \quad \boldsymbol{n}$$

$$\begin{aligned} &= k_1 \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \\ &+ k_2 \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \\ &\ \ \vdots \\ &+ k_m \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{aligned}$$

Column Spaces

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

COLUMN Space **subspace** of R^m

$$= span\{\boldsymbol{c_1}, \boldsymbol{c_2}, \cdots, \boldsymbol{c_n}\}$$

$$= \{\boldsymbol{w}\}$$

$$\boldsymbol{c}_i \in R^m \quad \begin{array}{cccc} \boldsymbol{c_1} & \boldsymbol{c_2} & & \boldsymbol{c_n} \\ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} & \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \end{array}$$

$\boldsymbol{m}$

$$\boldsymbol{w} = k_1\boldsymbol{c_1} + k_2\boldsymbol{c_2} + \cdots + k_n\boldsymbol{c_n}$$

$$= k_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + k_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots + k_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Null Space

$$\begin{array}{c} \\ m \end{array}\overbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}^{n} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} m$$

NULL Space

subspace of R^n

solution space

$$= \begin{pmatrix} a_{11}x_1 + & a_{12}x_2 + & \cdots & a_{1n}x_n \\ a_{21}x_1 + & a_{22}x_2 + & \cdots & a_{2n}x_n \\ \vdots & \vdots & & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \cdots & a_{mn}x_n \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$$\boldsymbol{A}\boldsymbol{x} = x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots + x_n\boldsymbol{c}_n = \boldsymbol{0} \qquad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$$

$$\boldsymbol{A}\boldsymbol{x} = x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots + x_n\boldsymbol{c}_n = \boldsymbol{b} \qquad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

Null Space

$$
m \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{\stackrel{\longleftrightarrow}{n}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} m
$$

NULL Space — **subspace** of R^n

solution space — $\mathbf{A\,x = 0}$

Invertible A — $\mathbf{x = A^{-1} 0 = 0}$ — only trivial solution — $\{\mathbf{0}\}$

Non-invertible A $\cancel{A^{-1}}$

zero row(s) in a RREF	free variables	parameters *s, t, u, …*	
one	one	a <u>line</u> through the origin	R^1
two	two	a <u>plane</u> through the origin	R^2
three	three	a <u>3-dim</u> space through the origin	R^3

Solution Space of **Ax=b** (1)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$$

$$\cancel{0 = 1}$$

$$1 \cdot x_1 + 3 \cdot x_3 = -1$$
$$1 \cdot x_2 - 4 \cdot x_3 = 2$$

$$1 \cdot x_1 - 5 \cdot x_2 + 1 \cdot x_3 = 4$$

Solve for a leading variable

$$x_1 = -1 - 3 \cdot x_3 \qquad\qquad x_1 = 4 + 5 \cdot x_2 - 1 \cdot x_3$$
$$x_2 = 2 + 4 \cdot x_3$$

Treat a free variable as a parameter

$$x_3 = t \qquad\qquad x_2 = s \quad x_3 = t$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

Solution Space of **Ax=b** (2)

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

$x_3 = t$ ⇐ free variable

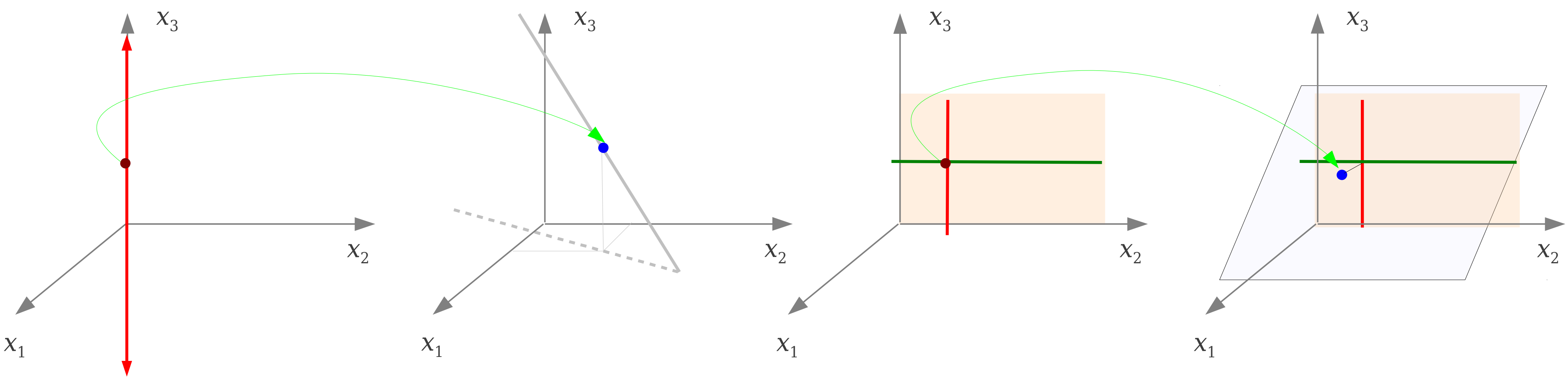
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

$x_2 = s$ ⇐ free variable

$x_3 = t$ ⇐ free variable

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$



infinitely many solutions *infinitely many solutions*

Solution Space of **Ax=b** (3)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}} + \boxed{t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}} + \boxed{s \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

General Solution of $\boldsymbol{Ax = b}$ ⇒

Particular Solution of $\boldsymbol{Ax = b}$

General Solution of $\boldsymbol{Ax = 0}$

Particular Solution of $\boldsymbol{Ax = b}$

General Solution of $\boldsymbol{Ax = 0}$

Linear System & Inner Product (1)

Linear Equations

Corresponding Homogeneous Equation

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

$$\mathbf{a} = (a_1 , a_2 , \cdots , a_n)$$

$$\mathbf{x} = (x_1 , x_2 , \cdots , x_n)$$

normal vector

$$\mathbf{a} \cdot \mathbf{x} = b$$

$$\mathbf{a} \cdot \mathbf{x} = 0$$

> each **solution** vector $\mathbf{x}$ of a **homogeneous** equation
> **orthogonal** to the coefficient vector $\mathbf{a}$

Homogeneous Linear System

$$\begin{array}{l} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = 0 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = 0 \\ \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = 0 \end{array}$$

$$\begin{array}{l} \mathbf{r_1} \cdot \mathbf{x} = 0 \\ \mathbf{r_2} \cdot \mathbf{x} = 0 \\ \quad \cdots \\ \mathbf{r_m} \cdot \mathbf{x} = 0 \end{array}$$

Linear System & Inner Product (2)

Homogeneous Linear System

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
\qquad
\begin{aligned}
\boldsymbol{r_1}\cdot\boldsymbol{x} &= 0 \\
\boldsymbol{r_2}\cdot\boldsymbol{x} &= 0 \\
\cdots & \\
\boldsymbol{r_m}\cdot\boldsymbol{x} &= 0
\end{aligned}
$$

each **solution** vector $\boldsymbol{x}$ of a **homogeneous** equation
orthogonal to the row vector $\boldsymbol{r_i}$ of the coefficient matrix

Homogeneous Linear System $\boldsymbol{A}\cdot\boldsymbol{x} = \boldsymbol{0}$ $\quad \boldsymbol{A} : m\times n$

solution set consists of all vectors in R^n
that are **orthogonal** to every row vector of $\boldsymbol{A}$

Linear System & Inner Product (3)

Non-Homogeneous Linear System $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$ $\quad \boldsymbol{A} : m \times n$

Homogeneous Linear System $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{0}$

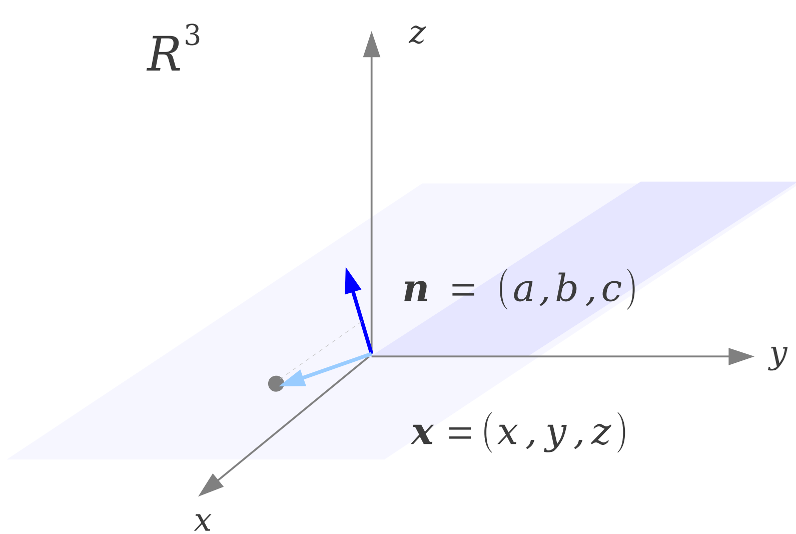
a particular solution

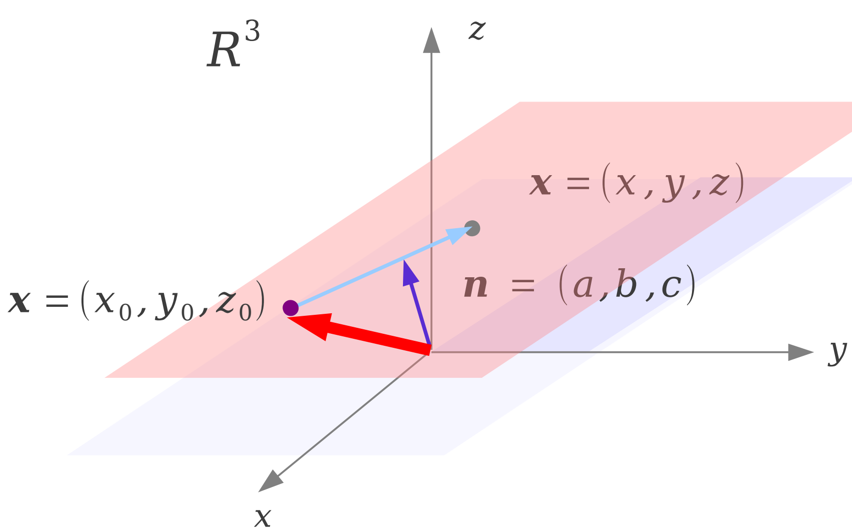
$\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$

solution set consists of all vectors in R^n that are **orthogonal** to every row vector of $\boldsymbol{A}$

+

a particular solution $\boldsymbol{x}_0$ $\quad \boldsymbol{A} \cdot \boldsymbol{x}_0 = \boldsymbol{b}$





Linear System & Inner Product (4)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$3 \begin{cases} 2 \begin{cases} \boldsymbol{r_1} \cdot \boldsymbol{x} = 0 \\ \boldsymbol{r_2} \cdot \boldsymbol{x} = 0 \end{cases} \\ 1 \begin{cases} \text{a line through the origin} \quad R^1 \end{cases} \end{cases} \qquad 3 \begin{cases} 1 \begin{cases} \boldsymbol{r_1} \cdot \boldsymbol{x} = 0 \end{cases} \\ 2 \begin{cases} \text{a plane through the origin} \quad R^2 \end{cases} \end{cases}$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

Consistent Linear System **Ax=b**

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + & a_{12}x_2 + & \cdots & a_{1n}x_n \\ a_{21}x_1 + & a_{22}x_2 + & \cdots & a_{2n}x_n \\ \vdots & \vdots & & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \cdots & a_{mn}x_n \end{pmatrix}$$

$$= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ consistent ⟺

$x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots + x_n\boldsymbol{c}_n = \boldsymbol{b}$

expressed in linear combination of column vectors

⟺ $\boldsymbol{b}$ is in the column space of $\boldsymbol{A}$

$$\boldsymbol{A}\boldsymbol{x} = x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots + x_n\boldsymbol{c}_n = \boldsymbol{b}$$

Dimension

In a finite-dimensional vector space R^n $\cancel{R^\infty}$

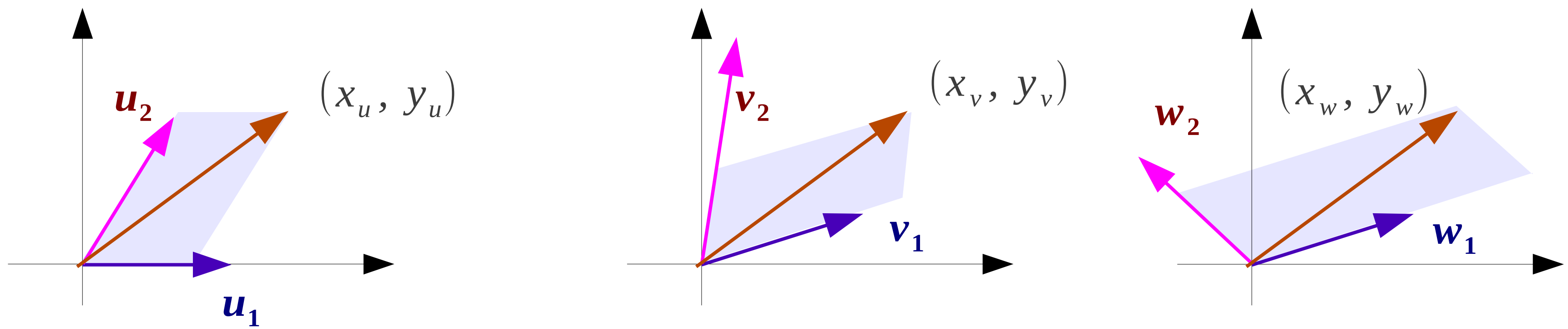
all bases ⇒ the same number of vectors n

many bases but the same number of basis vectors

basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ R^2

basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ R^2

basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ R^2



The dimension of a finite-dimensional vector space V

$\boxed{\text{dim(V)}}$ ⇔ the number of vectors in a basis

Dimension of a Basis (1)

In vector space R^2

	any one vector	(linearly indep.)	~~spans R^2~~	line <u>through **0**</u>
basis	any two non-collinear vectors	(linearly indep.)	spans R^2	plane
	any three or more vectors	~~(linearly indep.)~~	spans R^2	plane

In vector space R^3

	any one vector	(linearly indep.)	~~spans R^3~~	line <u>through **0**</u>
	any two non-collinear vectors	(linearly indep.)	~~spans R^3~~	plane <u>through **0**</u>
basis	any three vectors non-collinear, non-coplanar	(linearly indep.)	spans R^3	3-dim space
	any four or more vectors	~~(linearly indep.)~~	spans R^3	3-dim space

Dimension of a Basis (2)

In vector space R^n

	any n-1 vectors	(linearly indep.)?	~~spans~~	$\cancel{R^n}$	line through **0**
basis	n vectors of a basis	(linearly indep.)	**spans**	R^n	plane
	any n+1 vectors	~~(linearly indep.)~~	**spans?**	R^n	plane

a finite-dimensional vector space V

a basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$

- a set of more than n vectors ➡ ~~(linearly indep.)~~
- a set of less than n vectors ➡ ~~spans~~ $\cancel{V}$

$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ non-empty finite set of vectors in V

S is a basis ⟺
- S linearly independent
- S spans V

Basis Test

$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ non-empty finite set of vectors in V

S is a basis $\Leftrightarrow$ $\begin{cases} S \text{ linearly independent} \\ S \text{ spans } V \end{cases}$

V an **n**-dimensional vector space

$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ a set of **n** vectors in V

S linearly independent $\Rightarrow$ S is a basis

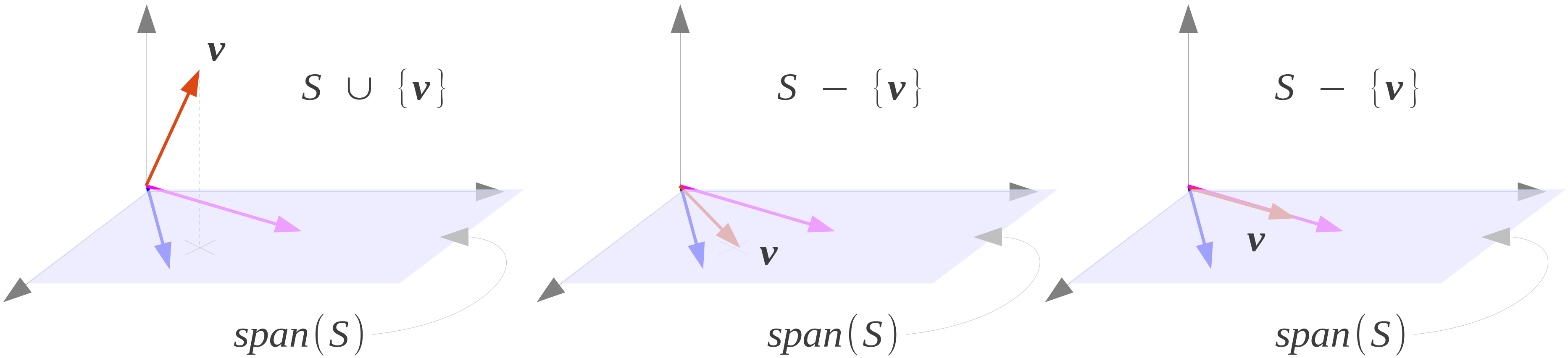
S spans V $\Rightarrow$ S is a basis

Plus / Minus Theorem

S a nonempty set of vectors in a vector space V

$\begin{cases} S & \text{: linear independent} \\ \boldsymbol{v} & \text{a vector in V but outside of span(S)} \end{cases}$ ⟹ $S \cup \{\boldsymbol{v}\}$: linear independent

$\begin{cases} \boldsymbol{v}, \ \boldsymbol{u_i} \in S & \text{linear combination} \\ \boldsymbol{v} = k_1 \boldsymbol{u_1} + k_2 \boldsymbol{u_2} + \cdots + k_n \boldsymbol{u_n} \end{cases}$ ⟹ $span(S) = span(S - \{\boldsymbol{v}\})$



Finding a Basis

S a nonempty set of vectors in a vector space V

$\begin{cases} S & \text{: linear independent} \\ \boldsymbol{v} & \text{a vector in V but outside of span(S)} \end{cases}$ ➡ $S \cup \{\boldsymbol{v}\}$: linear independent

if S is a *linearly independent* set that is *not already a basis* for V,
then S can be *enlarged* to a basis for V
by ***inserting*** appropriate vectors into S

$\begin{cases} \boldsymbol{v}\,,\ \boldsymbol{u_i} \in S & \text{linear combination} \\ \boldsymbol{v} = k_1\boldsymbol{u_1} + k_2\boldsymbol{u_2} + \cdots + k_n\boldsymbol{u_n} \end{cases}$ ➡ $span(S) = span(S - \{\boldsymbol{v}\})$

if S *spans* V but is *not a basis* for V,
then S can be *reduced* to a basis for V
by ***removing*** appropriate vectors from S

Vectors in a Vector Space

S a nonempty set of vectors in a vector space V

if S is a *linearly independent* set that is *not already a basis* for V,
then S can be *enlarged* to a basis for V
by ***inserting*** appropriate vectors into S

Every *linearly independent* set in a subspace is
either a **basis** for that subspace
or can be **extended to a basis** for it

if S *spans* V but is *not a basis* for V,
then S can be *reduced* to a basis for V
by ***removing*** appropriate vectors from S

Every *spanning set* for a subspace is
either a **basis** for that subspace
or has a **basis as a subset**

Dimension of a Subspace

W a subspace of a finite-dimensional vector space V

W is *finite-dimensional*

dim(W) ≤ dim(V)

W = V ⟺ dim(W) = dim(V)

Rank and Nullity

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

ROW Space **subspace** of R^n

$$= span\{\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_m\}$$

COLUMN Space **subspace** of R^m

$$= span\{\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_n\}$$

NULL Space **subspace** of R^n **solution space** $A\boldsymbol{x} = \boldsymbol{0}$

Invertible A $\boldsymbol{x} = A^{-1}\boldsymbol{0} = \boldsymbol{0}$ only trivial solution

Non-invertible A zero row(s) in a RREF free variables parameters *s, t, u, …*

dim(row space of A) = dim(column space of A) = **rank**(A)

dim(null space of A) = **nullity**(A)

Solution Space of **Ax=0**

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

the same case

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

General Solution of $Ax = \mathbf{0}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

dim(row space of A)
dim(col space of A)

rank(A) = 2 **rank**(A) = 1

dim(null space of A)

nullity(A) = 1 **nullity**(A) = 2

Elementary Row Operation (1)

ROW Space **subspace** of R^n

$= span\{\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_m\}$

COLUMN Space **subspace** of R^m

$= span\{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}$

NULL Space **subspace** of R^n

solution space $A\mathbf{x} = \mathbf{0}$

free variables parameters *s, t, u, …*

Elementary row operations do not change the **null space** of a matrix

Elementary row operations do not change the **row space** of a matrix

Elementary row operations ***do* change** the **col space** of a matrix

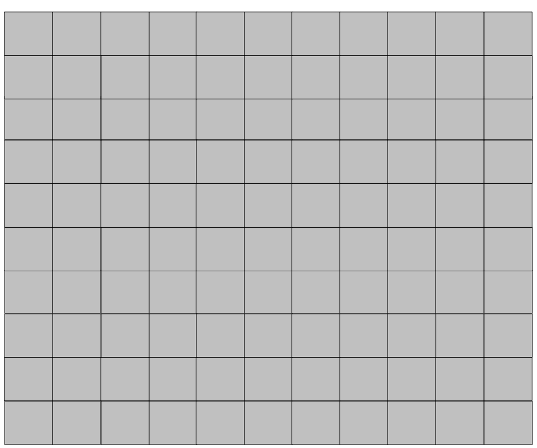
Elementary row operations do not change
the **linear dependence** and **linear independence** relationship
among column vectors

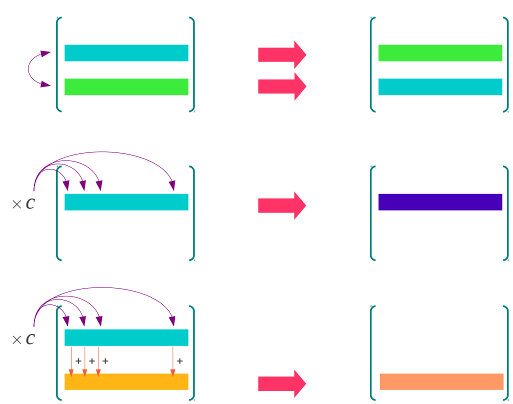
Elementary Row Operation (2)

Elementary row operations do not change the **null space** of a matrix
Elementary row operations do not change the **row space** of a matrix
Elementary row operations do not change the **linear dependence** and **linear independence** relationship among column vectors

Elementary row operations ***do* change** the **col space** of a matrix

A





R

1	0		0		0	0	0		0	
0	1		0		0	0	0		0	
0	0	0	1		0	0	0			
0	0	0	0	0	1	0	0		0	
0	0	0	0	0	0	1	0			
0	0	0	0	0	0	0	1		0	
0	0	0	0	0	0	0	0	0	1	
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

Elementary Row Operation (3)

Elementary row operations

- do not change the **null space** of a matrix
- do not change the **row space** of a matrix
- do not change the **linear dependence** and **linear independence** relationship among column vectors

- ***do* change** the **col space** of a matrix

A

row space of **A** = **row** space of **R**
null space of A = **null** space of **R**
col space of **A** ≠ **col** space of **R**

R

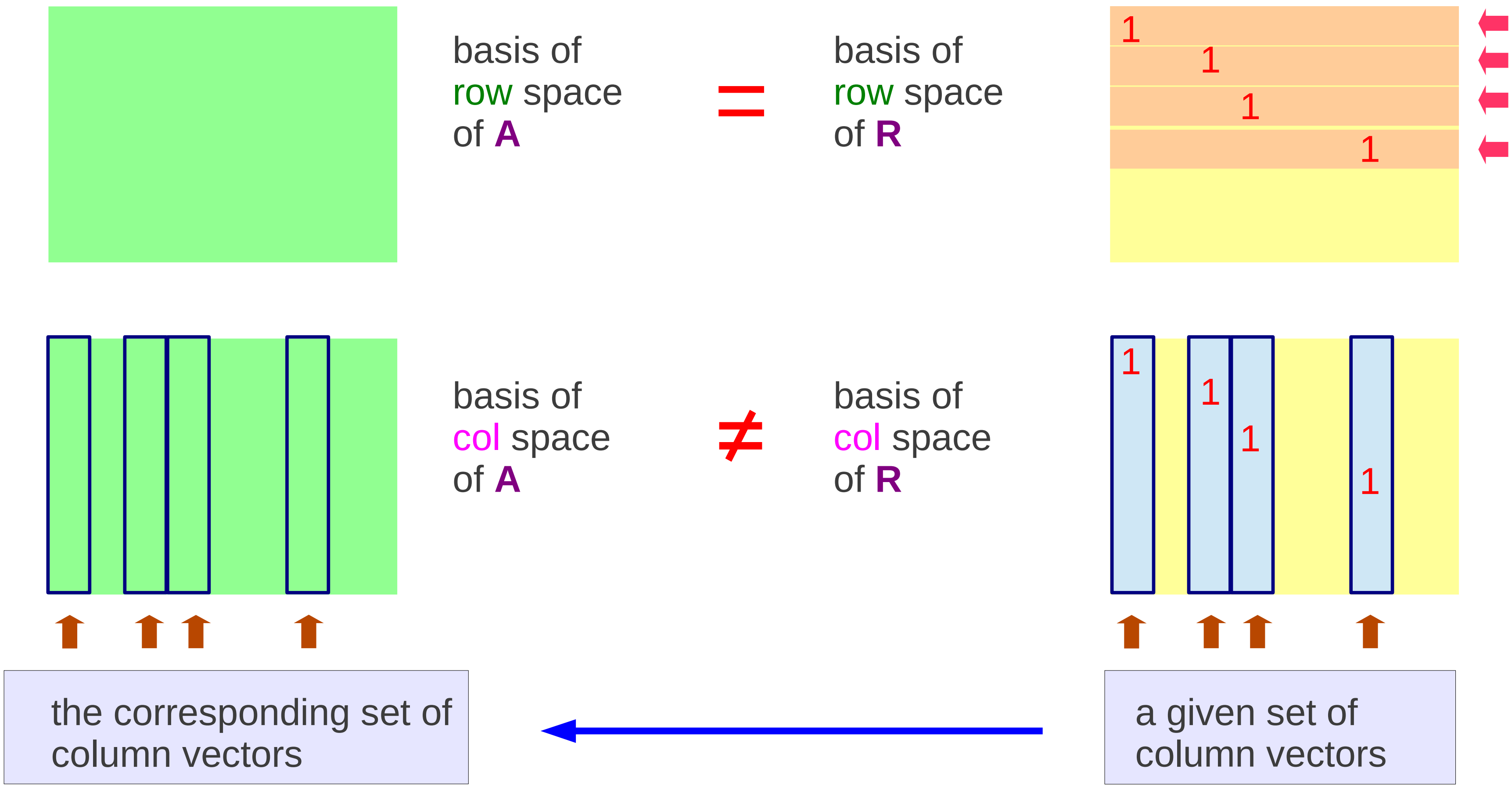
a given set of column vectors

linearly independent ⟺ linearly independent

basis ⟺ basis

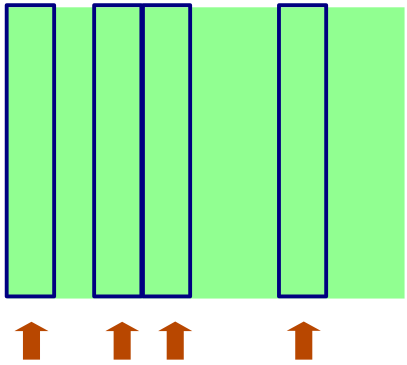
the corresponding set of column vectors

Bases of Row & Column Spaces (1)



dim(row space of A) = dim(column space of A) = **rank**(A)

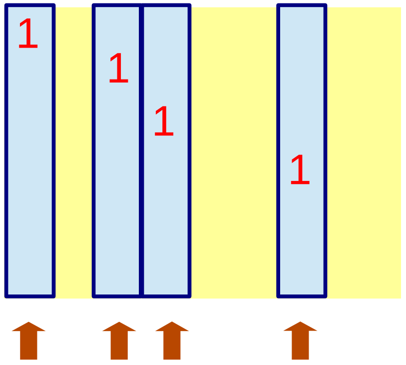
Bases of Row & Column Spaces (2)



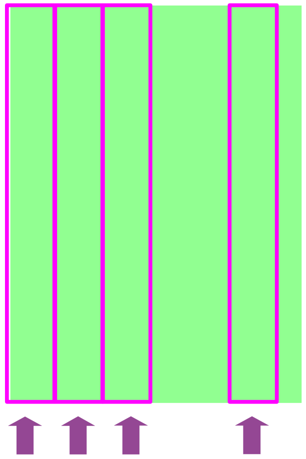
basis of
col space
of **A**

$\neq$

basis of
col space
of **R**



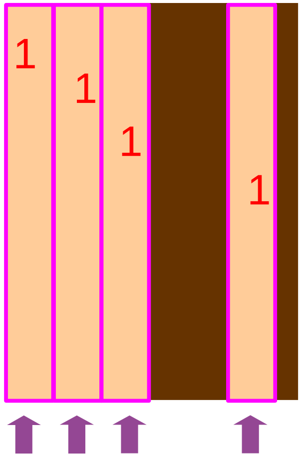
the basis consisting of columns of **A**



basis of
col space
of **A**

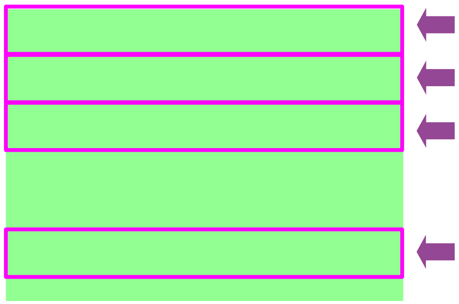
$\neq$

basis of
col space
of **R**

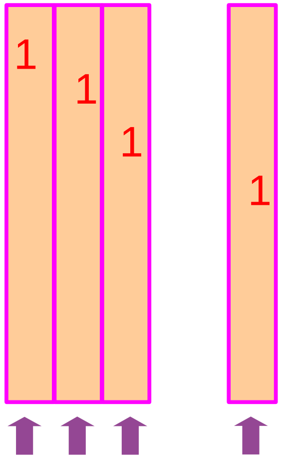


the basis consisting of rows of **A**

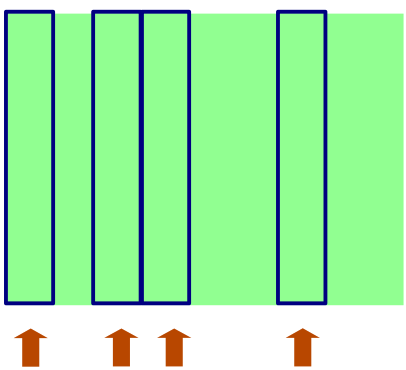
Bases of Row & Column Spaces (3)



basis of
col space
of **R**



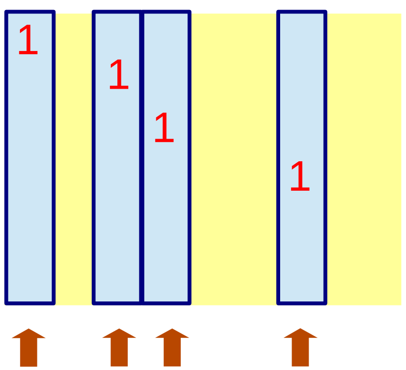
the basis consisting of rows of **A**



basis of
col space
of **A**

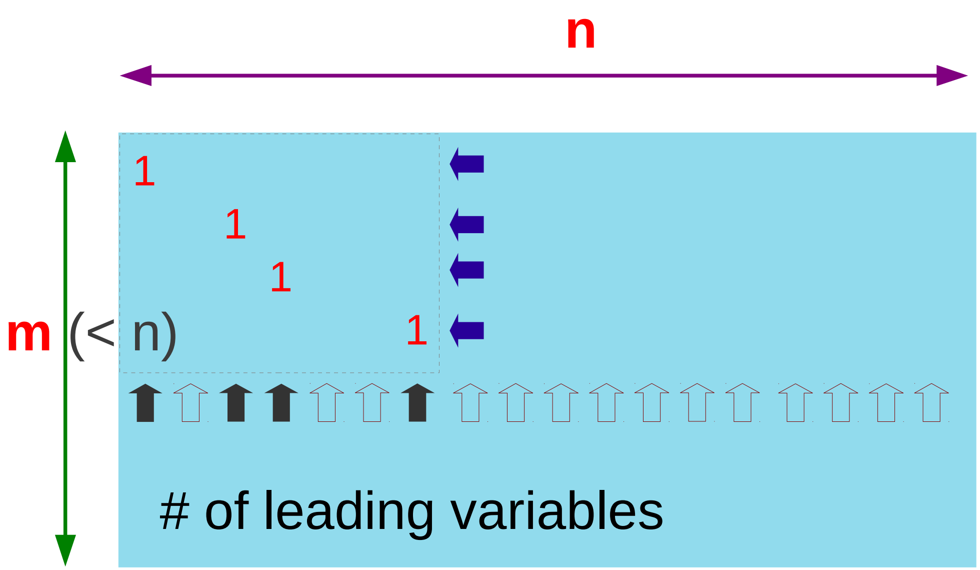
$\neq$

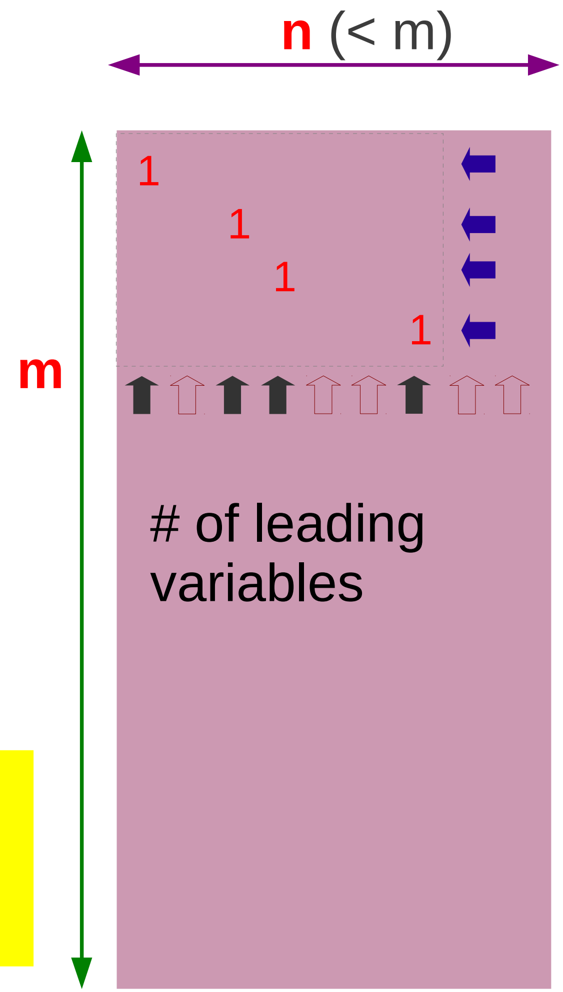
basis of
col space
of **R**

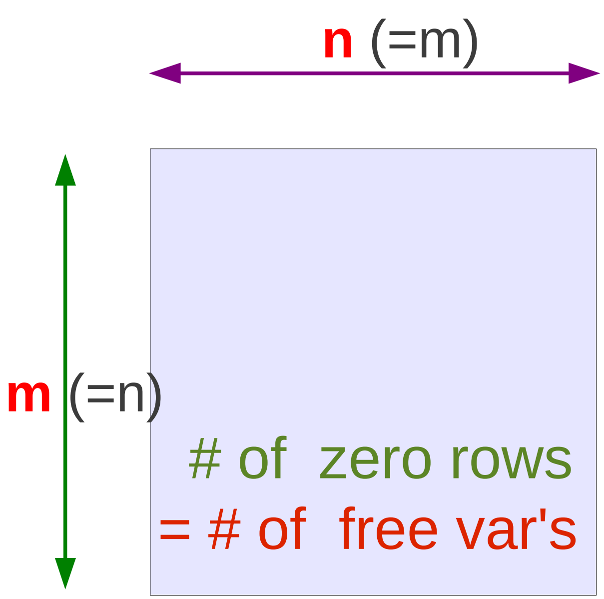


the basis consisting of columns of **A**

Rank and Nullity







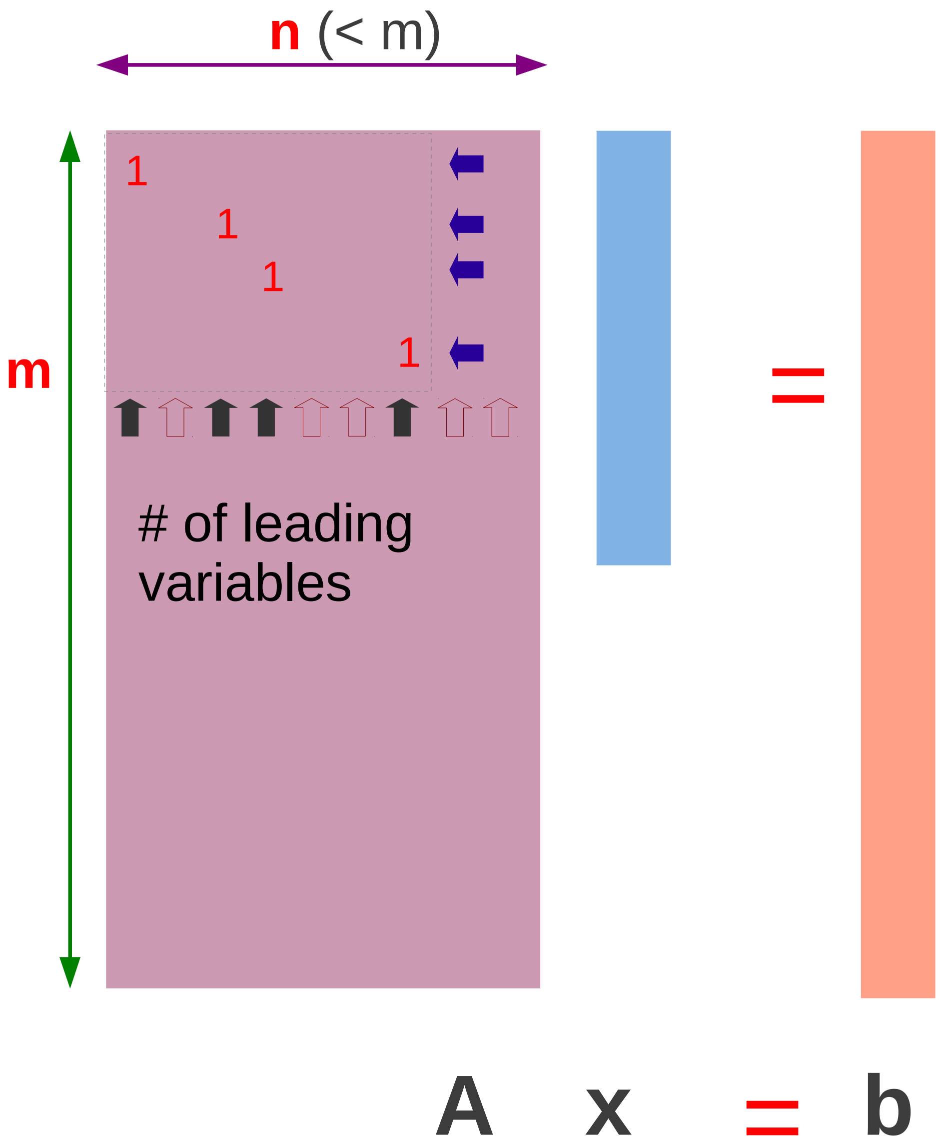
rank(A) < min(**m**, n)

n = **rank**(A) + **nullity**(A)

m = (# of leading variables) + (# of zero rows)

n = (# of leading variables) + (# of free variables)

Overdetermined System



n column vectors
can span at most R^n

b is in R^m $R^m \supset R^n$

At least one vector **b** in R^m
does not lie in column space

For such **b** in R^m
Ab = b inconsistent

References

[1] http://en.wikipedia.org/
[2] Anton, et al., Elementary Linear Algebra, 10th ed, Wiley, 2011
[3] Anton, et al., Contemporary Linear Algebra,