Minimum Phase (2A)

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7/21/12

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Properties of a Minimum Phase System

Lowest Time Delay

- Group Delay
- Energy Compaction

Invertible

- Min Phase Filter: { flat response / correct phase response }
- Equalizer: { flat response / incorrect phase response }

Minimum Phase System

Stable Causal System

All its poles are in the left half of the s plane

Minimum Phase System

All poles
All zeros

Maximum Phase System

All poles
All zeros

Mixed Phase System

All poles
some zeros
some zeros

Minimum Phase System Properties (1)

Minimum Phase System

Amplitude Response is known

$$A(\omega) = |H(j\omega)| \qquad 0 \le \omega < \infty$$

⬇

Minimum Phase Response can be computed

$$\Phi_{min}(\omega) = arg\{H(j\omega)\}$$

Non-Minimum Phase System

Poles / Zeros in the right half s plane

Amplitude Response is known

$$A(\omega) = |H(j\omega)| \qquad 0 \le \omega < \infty$$

Phase Response always greater

$$\Phi(\omega) \ge \Phi_{min}(\omega)$$

Minimum Phase System Properties (2)

Minimum Phase System		Non-Minimum Phase System
Phase Response $\Phi(\omega)$ can be unambiguously determined from the amplitude response $A(\omega)$	→	Not valid

Verification of a Minimum Phase System

Check the progression of $\Phi(\omega)$ and $A(\omega)$ at high frequency

$$H(\omega) = \frac{N(s)}{D(s)}$$

$N(s)$: order m

$D(s)$: order n

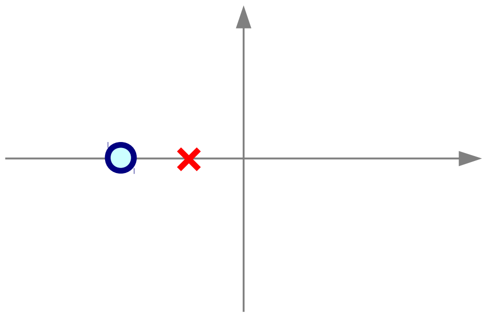
Minimum Phase System → $\Phi(\omega) = -90^\circ (n - m)$

Minimum Phase System
Non-Minimum → $-20(n - m) dB / decade$

Example

Minimum Phase System

$$H(s) = \frac{1 + 2s}{1 + 4s}$$



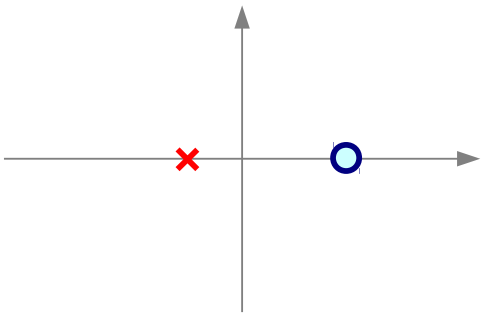
$$\frac{1 + j2\omega}{1 + j4\omega} = \frac{1 + j2\omega}{1 + j4\omega} \cdot \frac{1 - j4\omega}{1 - j4\omega}$$

$$= \frac{(1 + 8\omega^2) - j2\omega}{1 + 16\omega^2}$$

$$\Phi(\omega) = -\tan^{-1}\left(\frac{2\omega}{1 + 8\omega^2}\right)$$

Non-Minimum Phase System

$$H(s) = \frac{1 - 2s}{1 + 4s}$$



$$\frac{1 - j2\omega}{1 + j4\omega} = \frac{1 - j2\omega}{1 + j4\omega} \cdot \frac{1 - j4\omega}{1 - j4\omega}$$

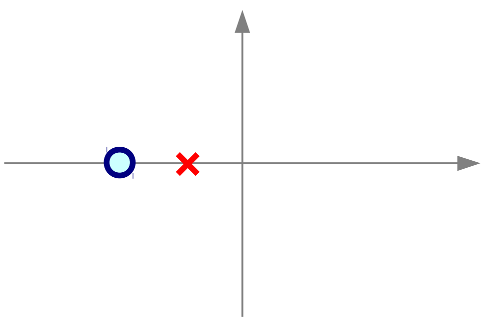
$$= \frac{(1 - 8\omega^2) - j6\omega}{1 + 16\omega^2}$$

$$\Phi(\omega) = -\tan^{-1}\left(\frac{6\omega}{1 + 8\omega^2}\right)$$

Non-minimum Phase System

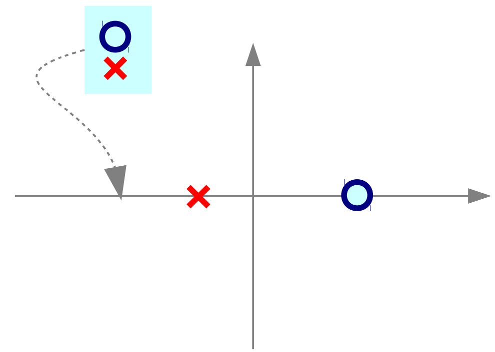
Minimum Phase System

$$H(s) = \frac{1 + 2s}{1 + 4s}$$



Non-Minimum Phase System

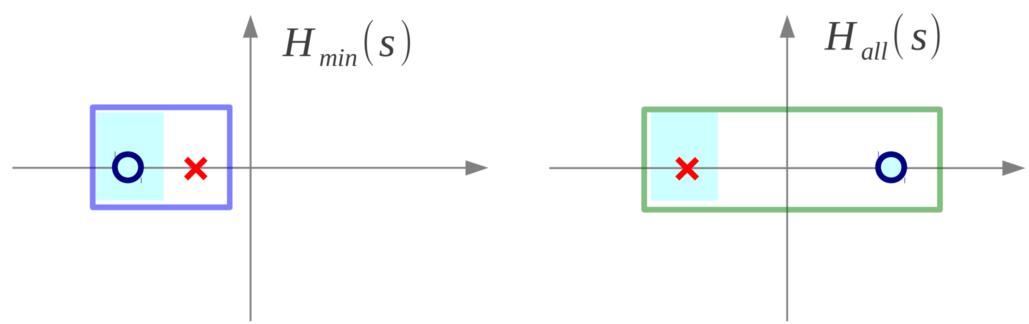
$$H(s) = \frac{1 - 2s}{1 + 4s} = \frac{1 - 2s}{1 + 4s} \cdot \frac{1 + 2s}{1 + 2s}$$



$$H(s) = \frac{1 - 2s}{1 + 4s} \cdot \frac{1 + 2s}{1 + 2s}$$

$$= \frac{1 + 2s}{1 + 4s} \cdot \frac{1 - 2s}{1 + 2s}$$

$$= H_{min}(s) \cdot H_{all}(s)$$



A non-minimum phase system can always be decomposed into $H_{min}(s) \cdot H_{all}(s)$

All Pass Filter (1)

$$H_{all}(s) = \frac{1-2s}{1+2s}$$

Flat Magnitude

$$\left|\frac{1-j2\omega}{1+j2\omega}\right| = \frac{|1-j2\omega|}{|1+j2\omega|}$$

$$= \frac{\sqrt{1+4\omega^2}}{\sqrt{1+4\omega^2}} = 1$$

$$|H_{all}(j\omega)| = \frac{1+4\omega^2}{1+4\omega^2} = 1$$

A Pure Phase Shifter

$$\frac{1-j2\omega}{1+j2\omega} = \frac{1-j2\omega}{1+j2\omega}\cdot\frac{1-j2\omega}{1-j2\omega}$$

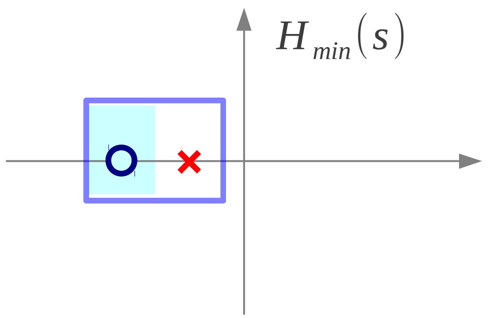
$$= \frac{(1-4\omega^2)-j4\omega}{1+4\omega^2}$$

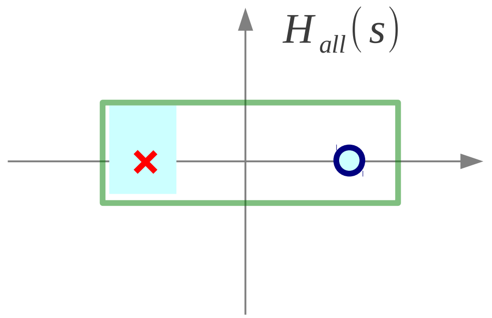
$$arg\{H_{all}(j\omega)\} = -\tan^{-1}\left(\frac{4\omega}{1-4\omega^2}\right)$$

$$H(s) = \frac{1-2s}{1+4s}\cdot\frac{1+2s}{1+2s}$$

$$= \frac{1+2s}{1+4s}\cdot\frac{1-2s}{1+2s}$$

$$= H_{min}(s)\cdot H_{all}(s)$$





A non-minimum phase system can always be decomposed into $H_{min}(s)\cdot H_{all}(s)$

All Pass Filter (2)

$$G_{all}(s) = \pm \frac{(s-\bar{s}_1)(s-\bar{s}_2)\ \cdots\ (s-\bar{s}_n)}{(s-s_1)(s-s_2)\ \cdots\ (s-s_n)}$$

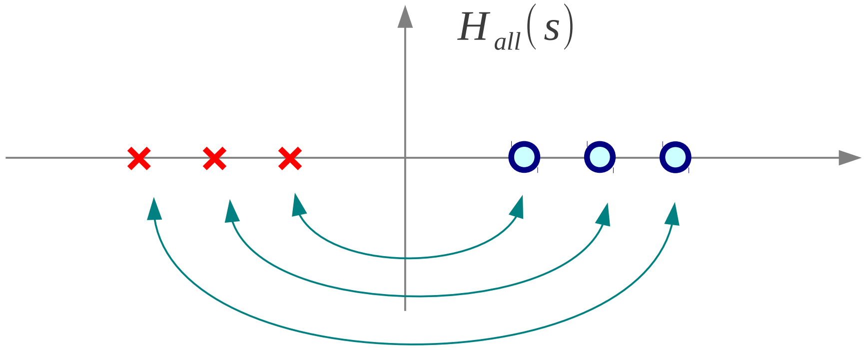
Flat Magnitude

A Pure Phase Shifter

zero $\bar{s}_i$

pole s_i

complex conjugate



All Pass Filter (3)

$$H_{all}(s) = \frac{s - 0.5}{s + 0.5}$$

$$= \frac{s + 0.5 - 1}{s + 0.5}$$

$$H(s) = 1 - \frac{2}{(s + 0.5)}$$

Inverse Laplace Transform

$$h(t) = \delta(t) - e^{-0.5t}$$

$$H_{all}(j\omega) = \frac{j\omega - 0.5}{j\omega + 0.5}$$

Flat Magnitude

$$|H_{all}(j\omega)| = \frac{\sqrt{\omega^2 + 0.25}}{\sqrt{\omega^2 + 0.25}} = 1$$

Phase Shifter

$$arg\{H_{all}(j\omega)\} = -2\tan^{-1}\left(\frac{\omega}{0.5}\right)$$

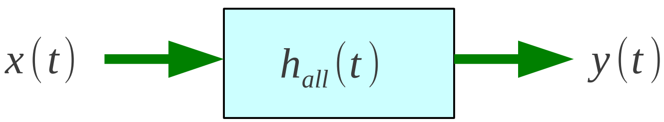
Group Delay

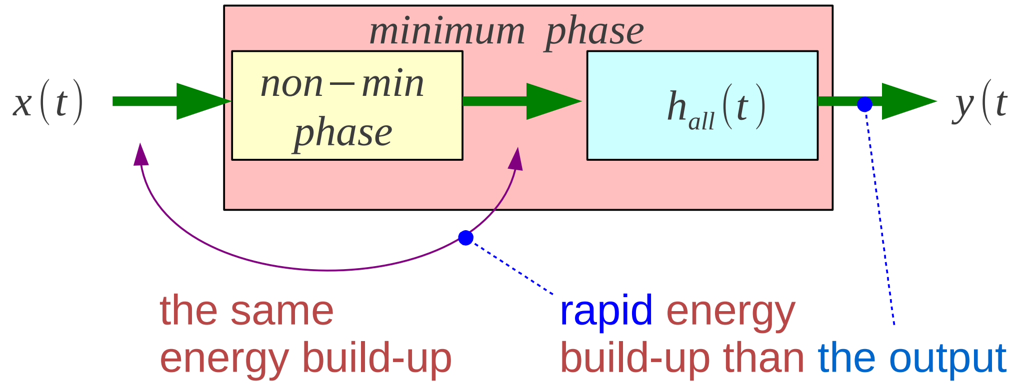
$$-\frac{d}{d\omega}\left(arg\{H_{all}(j\omega)\}\right)$$

$$= -\frac{d}{d\omega}\left(-2\tan^{-1}\left(\frac{\omega}{0.5}\right)\right)$$

$$= \frac{4}{(1 + \omega^2/0.25)} > 0$$

All Pass Filter (4)



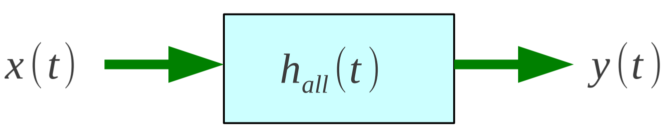


Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |y(t)|^2 \, dt$$

Energy Compaction

$$\int_{-\infty}^{t_0} |x(t)|^2 \, dt \geq \int_{-\infty}^{t_0} |y(t)|^2 \, dt$$



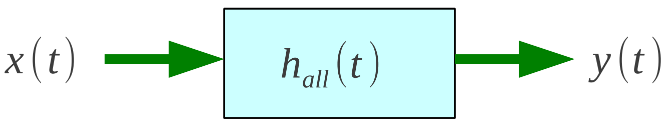
The energy build-up in the input
is more **rapid** than in the output

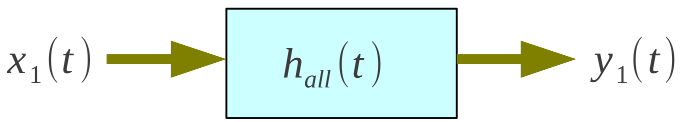
The signal energy until t_0 of the minimum phase
≥ any other causal signal
with the same magnitude response

Thus minimum phase signals
➡ maximally concentrated toward time 0
when compared against all causal signals
having the same magnitude response

minimum phase signals
➡ minimum delay signals

All Pass Filter (5)





Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |y(t)|^2 \, dt$$

$$x_1(t) \begin{cases} x(t) & (t \le t_0) \\ 0 & (t > t_0) \end{cases}$$

$$t \le t_0$$

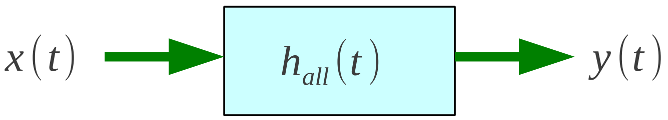
Energy Compaction

$$\int_{-\infty}^{t_0} |x(t)|^2 \, dt \ge \int_{-\infty}^{t_0} |y(t)|^2 \, dt$$

$$y_1(t) = \int_{-\infty}^{t_0} h(t-\tau) x_1(\tau) d\tau = \int_{-\infty}^{t} h(t-\tau) x(\tau) d\tau = y(t)$$

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |y(t)|^2 \, dt$$

$$\underline{\int_{-\infty}^{t_0} |x(t)|^2 \, dt} = \int_{-\infty}^{+\infty} |y(t)|^2 \, dt = \underline{\int_{-\infty}^{t_0} |y(t)|^2 \, dt} + \int_{t_0}^{+\infty} |y(t)|^2 \, dt$$



$$\int_{-\infty}^{t_0} |x(t)|^2 \, dt \ge \int_{-\infty}^{t_0} |y(t)|^2 \, dt$$

The energy build-up in the input is more **rapid** than in the output

Properties of a Minimum Phase System

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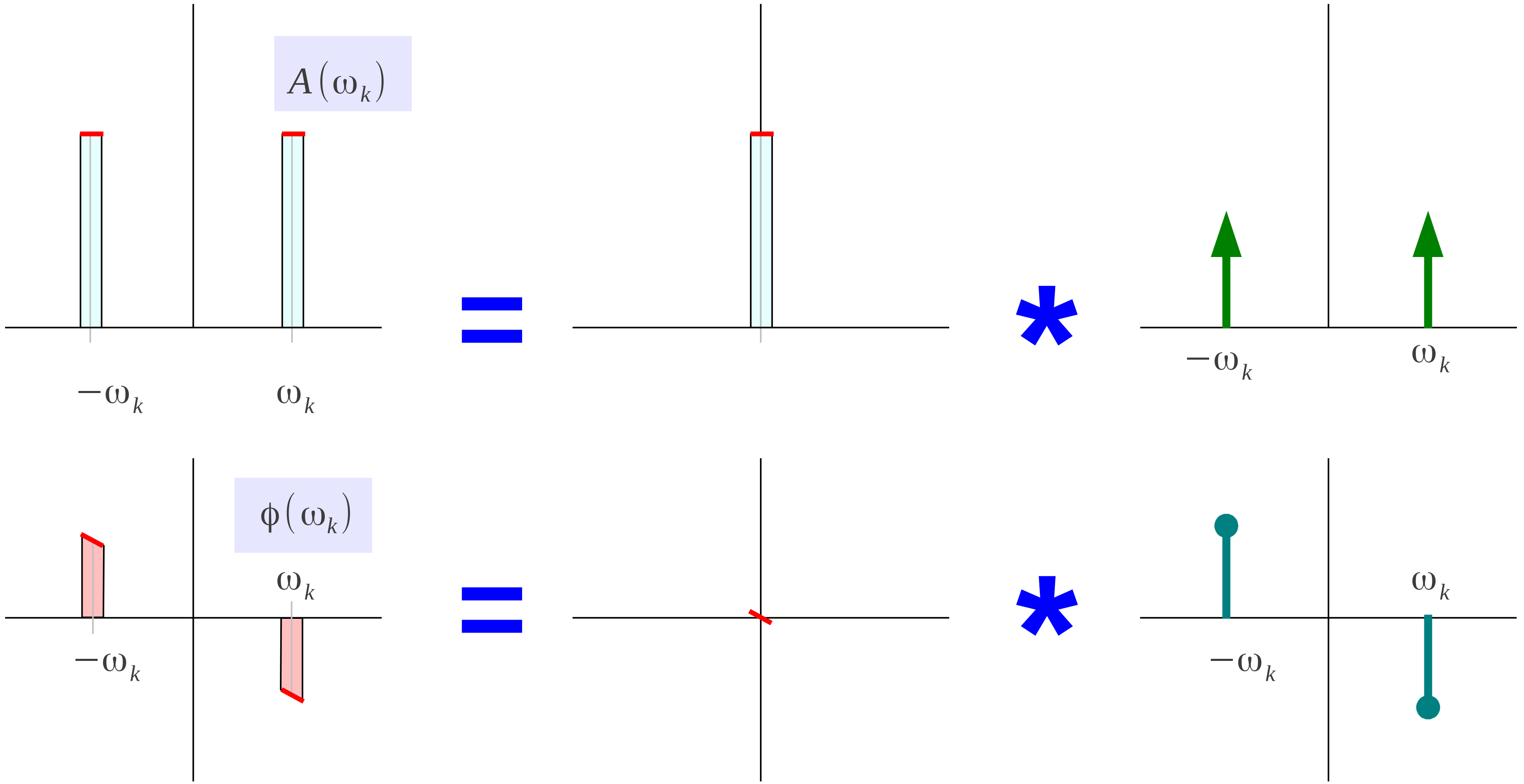
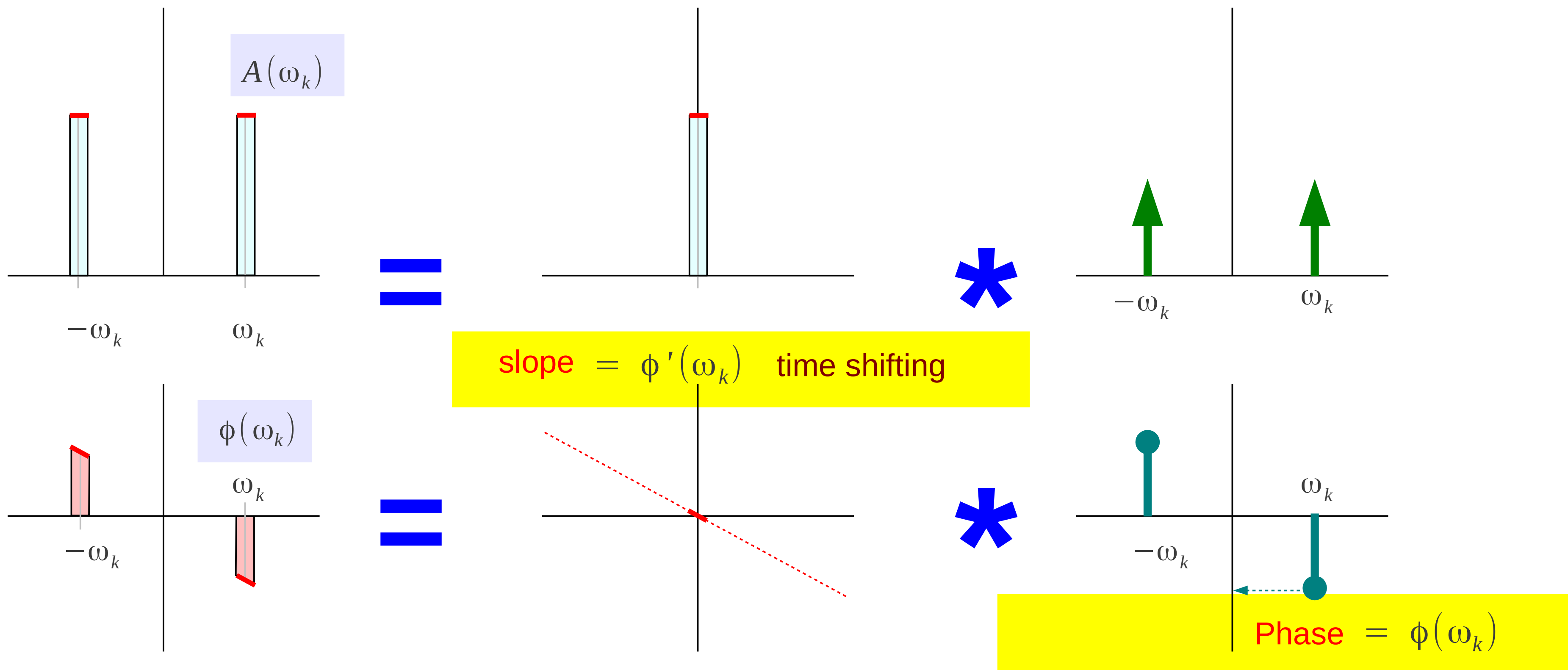
Simple LPF – Approximation (2)



$$2A(\omega_k)B\frac{\sin[B(t+\phi'(\omega_k))]}{\pi B(t+\phi'(\omega_k))} = \begin{cases} A(\omega_k) & (-B \le \omega \le +B) \\ 0 & else \end{cases} * \pi[e^{+j\phi(\omega_k)}\delta(\omega-\omega_k) + e^{-j\phi(\omega_k)}\delta(\omega+\omega_k)]$$

Simple LPF – Approximation (3)



Frequency Domain

$$\begin{cases} A(\omega_k) & (-B \le \omega \le +B) \\ 0 & else \end{cases} \quad * \quad \pi[e^{+j\phi(\omega_k)}\delta(\omega-\omega_k) + e^{-j\phi(\omega_k)}\delta(\omega+\omega_k)]$$

Group Delay

Phase Delay

Time Domain

$$2A(\omega_k)B\frac{\sin[B(t+\phi'(\omega_k))]}{\pi B(t+\phi'(\omega_k))} \quad \times \quad \cos(\omega_k t + \phi(\omega_k))$$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
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