2.7 Sturm Liouville.

We have worked on the following problem a lot.

$$(*)\quad \begin{bmatrix} \varphi'' + \lambda^2 \varphi = 0 \qquad l < x < r \\ \alpha_1 \varphi(l) - \alpha_2 \varphi'(l) = 0 \\ \beta_1 \varphi(r) - \beta_2 \varphi'(r) = 0 \end{bmatrix}$$

If $\varphi_n(x)$ and $\varphi_m(x)$ are solutions of $(*)$ with associated eigenvalues λ_n^2 and λ_m^2 then we can write:

$\varphi_n'' + \lambda_n^2 \varphi_n = 0$	$\varphi_m'' + \lambda_m^2 \varphi_m = 0$
$\varphi_m \varphi_n'' + \lambda_n^2 \varphi_m \varphi_n = 0$	$\varphi_n \varphi_m'' + \lambda_m^2 \varphi_n \varphi_m = 0$

The difference between the two equations

$$> \quad (\varphi_n \varphi_m'' + \lambda_m^2 \varphi_n \varphi_m) - (\varphi_m \varphi_n'' + \lambda_n^2 \varphi_m \varphi_n) = 0$$

$$> \quad (\varphi_n \varphi_m'' - \varphi_m \varphi_n'') + (\lambda_m^2 - \lambda_n^2) \varphi_n \varphi_m = 0$$

Integrate everything w.r.t x.

$$> \quad \int_l^r \varphi_n \varphi_m'' - \varphi_m \varphi_n'' \, dx + (\lambda_m^2 - \lambda_n^2) \int_l^r \varphi_n \varphi_m \, dx = 0$$

$$> \quad \int_l^r \varphi_n \varphi_m'' \, dx - \int_l^r \varphi_m \varphi_n'' \, dx + (\lambda_m^2 - \lambda_n^2) \int_l^r \varphi_n \varphi_m \, dx = 0$$

Integration by parts on first two terms.

$$\int_l^r \varphi_n \varphi_m'' \, dx = \varphi_n \varphi_m' \Big|_l^r - \int_l^r \varphi_m' \varphi_n' \, dx$$

$$\int_l^r \varphi_m \varphi_n'' \, dx = \varphi_m \varphi_n' \Big|_l^r - \int_l^r \varphi_m' \varphi_n' \, dx$$

Substitute into integral.

$$\varphi_n \varphi_m' \Big|_l^r - \int_l^r \varphi_m' \varphi_n' \, dx - \varphi_m \varphi_n' \Big|_l^r + \int_l^r \varphi_m' \varphi_n' \, dx$$

$$+ (\lambda_m^2 - \lambda_n^2) \int_l^r \varphi_n \varphi_m \, dx = 0$$

$$(\varphi_n \varphi_m' - \varphi_m \varphi_n')\Big|_l^r = (\lambda_n^2 - \lambda_m^2) \int_l^r \varphi_n \varphi_m \, dx$$

evaluate at end points

$$\underbrace{\left(\varphi_n(r)\varphi_m'(r) - \varphi_m(r)\varphi_n'(r)\right)}_{(\mathrm{I})} - \underbrace{\left(\varphi_n(l)\varphi_m'(l) - \varphi_m(l)\varphi_n'(l)\right)}_{(\mathrm{II})}$$

$$= (\lambda_n^2 - \lambda_m^2) \int_l^r \varphi_n \varphi_m \, dx$$

I and II turn out to be zero!

This follows on the next page.

From Boundary conditions.

for $\varphi_n(x)$ and $\varphi_m(x)$ we have.

> $\alpha_1 \varphi_n(\ell) - \alpha_2 \varphi_n'(\ell) = 0$

for both

> $\alpha_1 \varphi_m(\ell) - \alpha_2 \varphi_m'(\ell) = 0$

$$\begin{bmatrix} \varphi_n(\ell) & -\varphi_n'(\ell) \\ \varphi_m(\ell) & -\varphi_m'(\ell) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

system of equations

For a nontrivial solution $\alpha_1 = \alpha_2 = 0$

we need
$$\begin{vmatrix} \varphi_n(\ell) & -\varphi_n'(\ell) \\ \varphi_m(\ell) & -\varphi_m'(\ell) \end{vmatrix} = 0$$
"singular"

O.k.

> $- \varphi_n(\ell) \varphi_m'(\ell) + \varphi_n'(\ell) \varphi_m(\ell) = 0$

> $\varphi_n(\ell) \varphi_m'(\ell) - \varphi_n'(\ell) \varphi_m(\ell) = 0$

Term "II" from previous page must be zero.

Also from Boundary Conditions.

for $\varphi_n(x)$ and $\varphi_m(x)$

$$\beta_1 \varphi_n(r) + \beta_2 \varphi_n'(r) = 0$$

$$\beta_1 \varphi_m(r) + \beta_2 \varphi_m'(r) = 0$$

again $$\begin{bmatrix} \varphi_n(r) & \varphi_n'(r) \\ \varphi_m(r) & \varphi_m'(r) \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for nontrivial solution

$$\begin{vmatrix} \varphi_n(r) & \varphi_n'(r) \\ \varphi_m(r) & \varphi_m'(r) \end{vmatrix} = 0$$

$$\varphi_n(r)\, \varphi_m'(r) - \varphi_n'(r)\, \varphi_m(r) = 0$$

So term "I" must be zero.

Term I and II are zero so the eq.

$$> \overbrace{\left[\varphi_n(r)\varphi_m'(r) - \varphi_m(r)\varphi_n'(r)\right]}^{0} - \overbrace{\left[\varphi_n(l)\varphi_m'(l) - \varphi_m(l)\varphi_n'(l)\right]}^{0}$$

$$= (\lambda_n^2 - \lambda_m^2) \int_l^r \varphi_n \varphi_m \, dx$$

$$> \quad 0 = (\lambda_n^2 - \lambda_m^2) \int_l^r \varphi_n \varphi_m \, dx$$

$$\lambda_n^2 \neq \lambda_m^2$$

$$\boxed{0 = \int_l^r \varphi_n \varphi_m \, dx} \leftarrow \text{"BIG Result"}$$

solutions of $\varphi'' + \lambda^2 \varphi = 0$

with $\quad \alpha_1 \varphi(l) - \alpha_2 \varphi'(l) = 0$

$\qquad\qquad \beta_1 \varphi(r) + \beta_2 \varphi'(r) = 0$

are "orthogonal."

A more general problem of the same form.

Regular Sturm Liouville Problem

$$\left(s(x)\,\phi'(x)\right)' - q(x)\,\phi(x) + \lambda^2 p(x)\,\phi(x) = 0$$

$$\alpha_1 \phi(l) - \alpha_2 \phi'(l) = 0 \qquad l \le x \le r$$

$$\beta_1 \phi(r) - \beta_2 \phi'(r) = 0$$

Ex $\quad s(x) = 1 \quad p(x) = 1 \quad q(x) = 0$

$$\phi''(x) + \lambda^2 \phi(x) = 0 \qquad \phi(l) = 0,\ \phi'(r) = 0$$

- $s(x), s'(x), q(x), p(x)$ continuous on $l \le x \le r$
- $s(x) > 0$, $p(x) > 0$ on $l \le x \le r$
- $\alpha_1, \alpha_2 \ge 0 \quad \beta_1, \beta_2 \ge 0$, $\quad \alpha_1^2 + \alpha_2^2 > 0$, $\quad \beta_1^2 + \beta_2^2 > 0$

The regular sturm liouville problem has an infinite number of eigenfunctions $\phi_1, \phi_2, \phi_3, \ldots$ with different eigenvalues $\lambda_1^2, \lambda_2^2, \ldots$.

If $n \ne m$ we have $\displaystyle\int_l^r \phi_n(x)\,\phi_m(x)\,p(x)\,dx = 0$

$p(x)$ is a weighting function.

Ex For $s(x) = x \quad p(x) = \frac{1}{x} \quad q(x) = 0$

and $\varphi(1) = 0 \quad \varphi(2) = 0$

$$\varphi'' + \lambda^2 \left(\frac{1}{x}\right) \varphi(x) = 0 \qquad 1 \le x \le 2$$

the solutions are

$$\varphi(x) = c_1 \sin(\lambda \ln(x)) + c_2 \cos(\lambda(\ln(x)))$$

$$\varphi(1) = 0 = c_2$$

$$\varphi(2) = 0 = c_1 \sin(\lambda \ln(2))$$

assume $c_1 \neq 0$

$$0 = \sin(\lambda \ln(2))$$

$$\lambda_n \ln(2) = n\pi$$

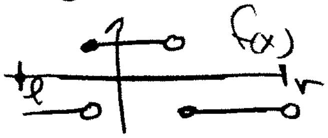
$$\boxed{\lambda_n = \frac{n\pi}{\ln(2)}}$$

$$\varphi_n(x) = \sin(\lambda_n \ln(x)) \qquad \lambda_n^2 = \left(\frac{n\pi}{\ln(2)}\right)^2$$

eigenfunction $\qquad$ eigenvalue.

f(x) is a piecewise continuous function.

For φ_n's and λ_n^2's the series $\sum_{n=1}^{\infty} c_n \varphi_n(x)$ converges at each $l \le x \le r$.



midpoint

$$\sum_{n=1}^{\infty} c_n \varphi_n(x) = \frac{f(x^-) + f(x^+)}{2}$$

where

$$c_n = \frac{\int_l^r \varphi_n(x)\, \varphi_m(x)\, dx}{\int_l^r \varphi_n^2\, dx}$$

We have been using all of this to solve our heat equation problems. Now you know why we need homogeneous B.C.s and that the solution are guaranteed to be orthogonal. Also we know that we can calculate (c_n's) with the above formula.