

CORDIC in VHDL (1A)

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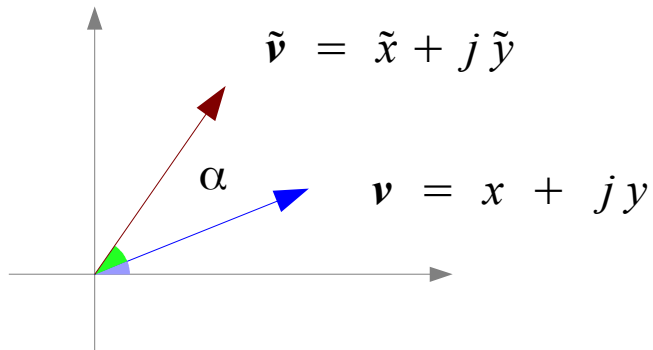
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CORDIC Background

1. G Hampson,
A VHDL Implementation of a CORDIC Arithmetic Processor Chip
Monash University, Technical Report 94-9, 1994

Angle Expansion

$$\tilde{\mathbf{v}} = \mathbf{v} e^{j\alpha}$$



elementary angle

$$\theta_0 = \tan^{-1}(2^0) =$$

$$\theta_1 = \tan^{-1}(2^{-1}) =$$

$$\theta_2 = \tan^{-1}(2^{-2}) =$$

$$\theta_3 = \tan^{-1}(2^{-3}) =$$

α can be expanded by
a set of elementary angles α_i
pseudo-digits q_i

$$\alpha_i \begin{cases} \pi/2 & i = -1 \\ \tan^{-1}(2^{-i}) & i = 0, 1, 2, \dots, n-1 \end{cases}$$

$$q_i \begin{cases} -1 \\ +1 \end{cases}$$

$$\alpha = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n$$

angle expansion error z_n

$$|z_n| \leq 2^{-(n-1)}$$

Rotating Vector

$$\tilde{\mathbf{v}} = \mathbf{v} e^{j\alpha}$$

$$= \mathbf{v} \exp\left(j\left(\sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n\right)\right)$$

$$= \mathbf{v} \cdot \left(\prod_{i=-1}^{n-1} e^{jq_i \alpha_i}\right) \cdot e^{jz_n}$$

$$= \mathbf{v} \cdot (jq_{-1}) \left(\prod_{i=0}^{n-1} e^{jq_i \alpha_i}\right) \cdot e^{jz_n}$$

$$= \mathbf{v} \cdot (jq_{-1}) \left(\prod_{i=0}^{n-1} \cos(\alpha_i) \cdot (1 + jq_i 2^{-i})\right) \cdot e^{jz_n}$$

$$= \mathbf{v} \cdot (jq_{-1}) \left(\prod_{i=0}^{n-1} \cos(\alpha_i)\right) \cdot \left(\prod_{i=0}^{n-1} (1 + jq_i 2^{-i})\right) \cdot e^{jz_n}$$

$$\alpha = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{jq_{-1}\alpha_0} = e^{jq_{-1}\frac{\pi}{2}} = jq_{-1} \quad (e^{\pm j\frac{\pi}{2}} = \pm j)$$

$$\begin{aligned} e^{jq_i \alpha_i} &= \cos(q_i \alpha_i) + j\sin(q_i \alpha_i) \\ &= \cos(q_i \alpha_i) \cdot (1 + j\tan(q_i \alpha_i)) \\ &= \cos(q_i \alpha_i) \cdot (1 + jq_i 2^{-i}) \\ &= \cos(\alpha_i) \cdot (1 + jq_i 2^{-i}) \end{aligned}$$

$$(\cos(\pm\alpha_i) = \cos(\alpha_i))$$



Rotating Vector

$$\tilde{\mathbf{v}} = \mathbf{v} e^{j\alpha} \quad \text{rotated by } \alpha$$

$$\alpha = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n$$

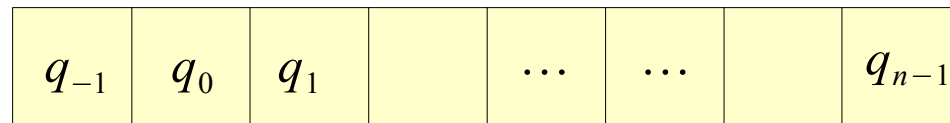
$$= \mathbf{v} \cdot \left(\prod_{i=0}^{n-1} \cos(\alpha_i) \right) \cdot (jq_{-1}) \cdot \left(\prod_{i=0}^{n-1} (1 + jq_i 2^{-i}) \right) \cdot e^{jz_n}$$



$$K_n = \prod_{i=0}^{n-1} \frac{1}{\sqrt{1 + 2^{-2i}}}$$

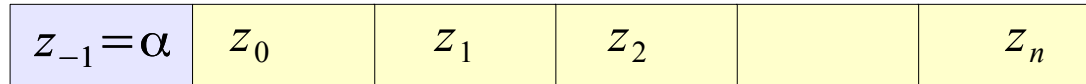
series rotations of α_i

n iterations



Rotating Vector

$$((((\alpha - q_{-1}\alpha_{-1}) - q_0\alpha_0) - q_1\alpha_1) \cdots - q_{n-1}\alpha_{n-1})$$



$$z_{i+1} = z_i - q_i \alpha_i$$



$$\begin{aligned} \text{if } (z_0 \geq 0) \quad q_0 &= +1 \\ \text{if } (z_0 < 0) \quad q_0 &= -1 \end{aligned}$$

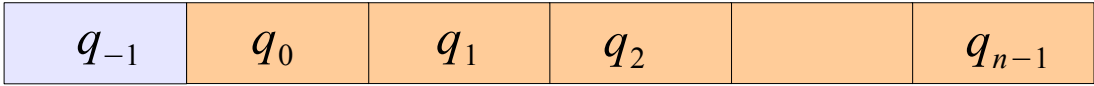
$$\begin{aligned} \text{if } (z_1 \geq 0) \quad q_1 &= +1 \\ \text{if } (z_1 < 0) \quad q_1 &= -1 \end{aligned}$$

$$\begin{aligned} \text{if } (z_2 \geq 0) \quad q_2 &= +1 \\ \text{if } (z_2 < 0) \quad q_2 &= -1 \end{aligned}$$

$$\begin{aligned} \text{if } (z_i \geq 0) \quad q_i &= +1 \\ \text{if } (z_i < 0) \quad q_i &= -1 \end{aligned}$$

$$\begin{aligned} \text{if } (z_{-1} \geq 0) \quad q_{-1} &= +1 \\ \text{if } (z_{-1} < 0) \quad q_{-1} &= -1 \end{aligned}$$

$$\begin{aligned} \text{if } (z_{n-1} \geq 0) \quad q_{n-1} &= +1 \\ \text{if } (z_{n-1} < 0) \quad q_{n-1} &= -1 \end{aligned}$$



Rotating Vector

$$\tilde{\mathbf{v}} = \mathbf{v} e^{j\alpha} \quad \text{rotated by } \alpha$$

$$\alpha = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n$$

Rotated vector after n iteration

$$\mathbf{v}_n = \mathbf{v} \cdot e^{j\alpha} \cdot e^{-jz_n}$$

$$z_n = \alpha - \sum_{i=-1}^{n-1} q_i \cdot \alpha_i$$

$$\mathbf{v}_{i+1} = \mathbf{v}_i \cdot (1 + jq_i \cdot 2^{-i})$$

$$\mathbf{v}_0 = \mathbf{v}_{-1} \cdot (jq_{-1})$$

$$\begin{aligned} x_{i+1} + jy_{i+1} &= (x_i + jy_i) \cdot (1 + jq_i \cdot 2^{-i}) \\ &= (x_i - y_i \cdot q_i \cdot 2^{-i}) + j(y_i + x_i \cdot q_i \cdot 2^{-i}) \end{aligned}$$

$$\begin{aligned} x_0 + jy_0 &= (x_{-1} + jy_{-1}) \cdot (jq_{-1}) \\ &= (-q_{-1} \cdot y_{-1}) + j(q_{-1} \cdot x_{-1}) \end{aligned}$$

$$\begin{cases} x_{i+1} = (x_i - y_i \cdot q_i \cdot 2^{-i}) \\ y_{i+1} = (y_i + x_i \cdot q_i \cdot 2^{-i}) \end{cases}$$

$$\begin{cases} x_0 = -q_{-1} \cdot y_{-1} \\ y_0 = q_{-1} \cdot x_{-1} \end{cases}$$

Rotating Vector

$$\tilde{\mathbf{v}} = \mathbf{v} e^{j\alpha} \quad \text{rotated by } \alpha$$

$$\alpha = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n$$

$$= \mathbf{v} \cdot \left(\prod_{i=0}^{n-1} \cos(\alpha_i) \right) \cdot (jq_{-1}) \cdot \left(\prod_{i=0}^{n-1} (1 + jq_i 2^{-i}) \right) \cdot e^{jz_n}$$

q_{-1}	q_0	q_1			q_{n-1}
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Rotating Vector

$$\begin{aligned} \text{if } (z_i \geq 0) & \quad q_i = +1 \\ \text{if } (z_i < 0) & \quad q_i = -1 \end{aligned}$$

$$v_{i+1} = v_i \cdot (1 + jq_i \cdot 2^{-i})$$



$$\begin{cases} x_{i+1} = (x_i - y_i \cdot q_i \cdot 2^{-i}) \\ y_{i+1} = (y_i + x_i \cdot q_i \cdot 2^{-i}) \end{cases}$$

$$z_{i+1} = z_i - q_i \alpha_i$$

$$v_0 = v_{-1} \cdot (jq_{-1})$$



$$\begin{cases} x_0 = -q_{-1} \cdot y_{-1} \\ y_0 = q_{-1} \cdot x_{-1} \end{cases}$$

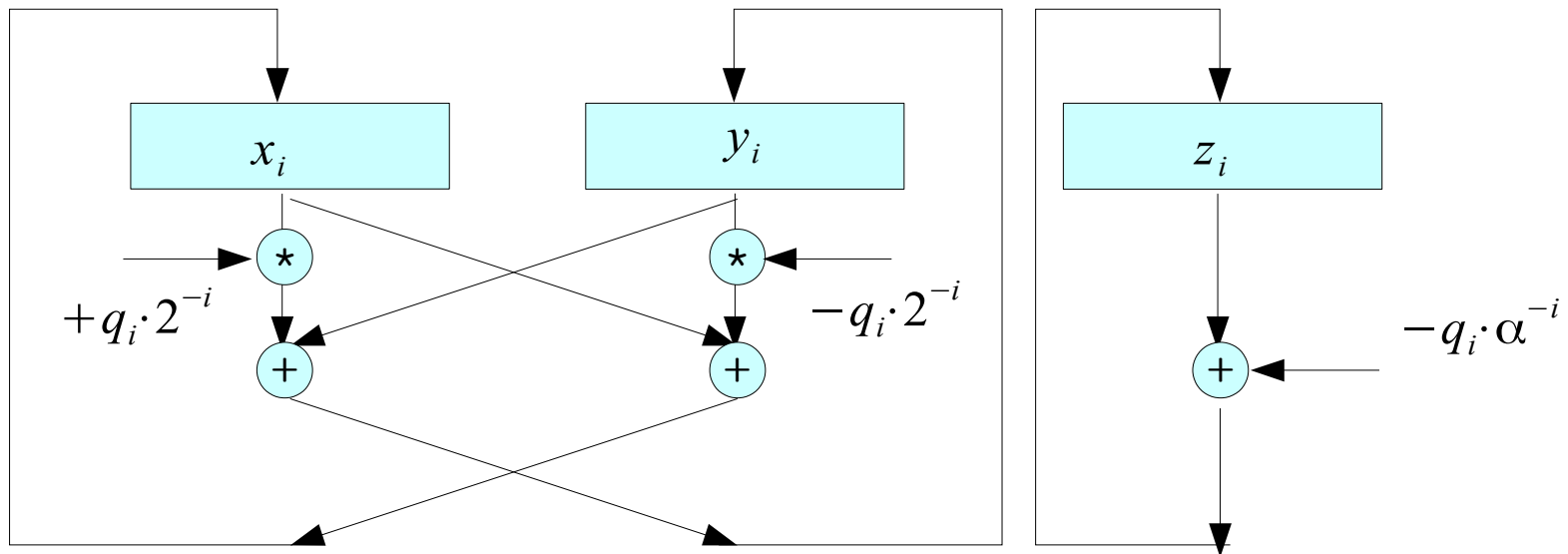
Rotating Vector

$$\begin{cases} \text{if } (z_i \geq 0) & q_i = +1 \\ \text{if } (z_i < 0) & q_i = -1 \end{cases}$$

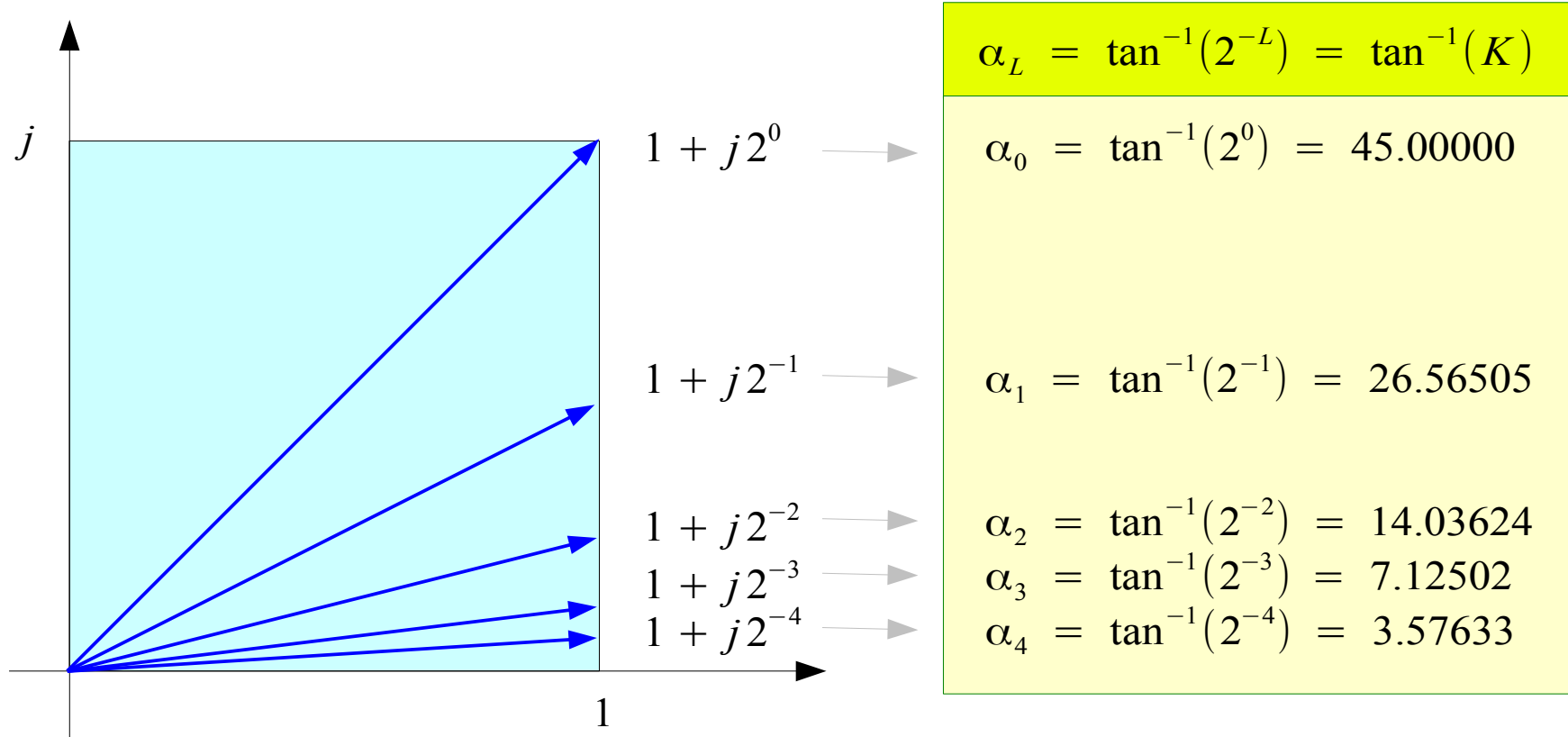
$$\begin{cases} x_{i+1} = (x_i - y_i \cdot q_i \cdot 2^{-i}) \\ y_{i+1} = (y_i + x_i \cdot q_i \cdot 2^{-i}) \end{cases}$$

$$\begin{cases} x_0 = -q_{-1} \cdot y_{-1} \\ y_0 = q_{-1} \cdot x_{-1} \end{cases}$$

$$z_{i+1} = z_i - q_i \alpha_i$$



Elementary Angle: $\tan^{-1}(K)$



Represent arbitrary angle α

in terms of $\pm\alpha_0, \pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \dots, \pm\alpha_L, \dots$ $\left(K = \frac{1}{2^L}, L = 0, 1, 2, \dots \right)$

Phase and Magnitude of $1 + jK$ (1)

Cumulative Magnitude

L	$K = \frac{1}{2^L}$	$R = 1 + jK$	Phase of R	Magnitude of R	CORDIC Gain
0	1.0	$1 + j1.0$	45°	1.41421356	1.414213562
1	0.5	$1 + j0.5$	26.56505°	1.11803399	1.581138830
2	0.25	$1 + j0.25$	14.03624°	1.03077641	1.629800601
3	0.125	$1 + j0.125$	7.12502°	1.00778222	1.642484066
4	0.0625	$1 + j0.0625$	3.57633°	1.00195122	1.645688916
5	0.03125	$1 + j0.03125$	1.78991°	1.00048816	1.646492279
6	0.015625	$1 + j0.015625$	0.89517°	1.00012206	1.646693254
7	0.007813	$1 + j0.007813$	0.44761°	1.00003052	1.646743507
...
					1.647 ←

$$R = 1 + jK \xrightarrow[L = 0, 1, 2, \dots]{K = 1/2^L} \sqrt{1^2 + K^2} > 1.0$$

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References

- [1] <http://en.wikipedia.org/>
- [2] G Hampson, A VHDL Implementation of a CORDIC Arithmetic Processor Chip
Monash University, Technical Report 94-9, 1994