

Complex Integration (2A)

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Contour Integrals

$f(z)$ defined at points
of a smooth curve C

a smooth curve C is defined by

$$\begin{aligned}x &= x(t) \\y &= y(t)\end{aligned} \quad a \leq t \leq b$$

The contour integral of f along C

$$\begin{aligned}\int_C f(z) dz &= \int_C (u + iv)(dx + idy) = \int_C \underline{u dx} - \underline{v dy} + i \int_C \underline{v dx} + \underline{u dy} \\&= \int_a^b [\underline{u x'(t)} - \underline{v y'(t)}] dt + i \int_a^b [\underline{v x'(t)} + \underline{u y'(t)}] dt \\&= \int_a^b (u + iv)(x'(t) + iy'(t)) dt\end{aligned}$$

$$\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt$$

$$z(t) = x(t) + iy(t)$$

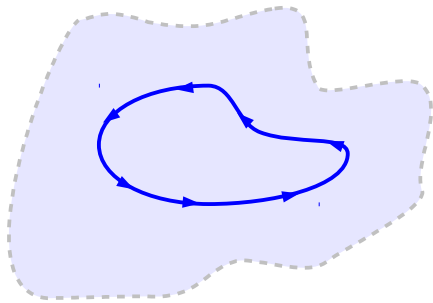
$$z'(t) = x'(t) + iy'(t)$$

$$a \leq t \leq b$$

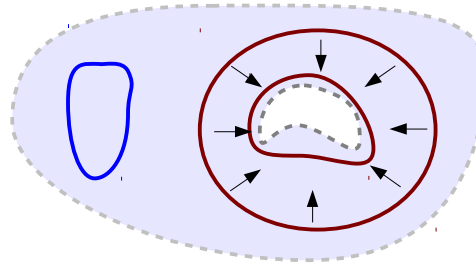
Connected Region

Connected Domains

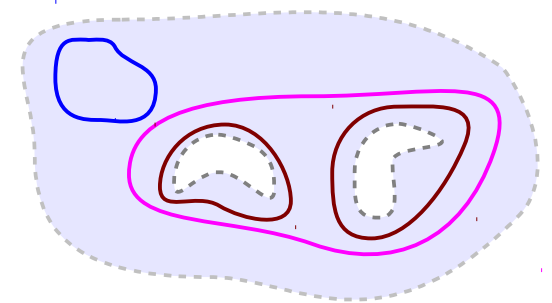
Simply Connected



Doubly Connected

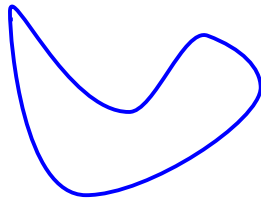


Triply Connected

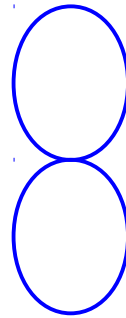


Closed Paths

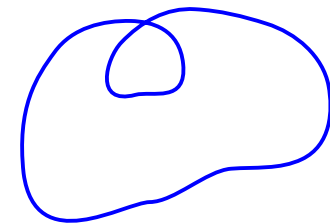
simple closed path



not simple closed path



not simple closed path



Cauchy's Theorem

$f(z)$: **analytic** in a **simply connected domain** D

$f'(z)$: **continuous** in a **simply connected domain** D



for every **simple closed contour** C in D

$$\oint_C f(z) dz = 0$$

$$\int_C f(z) dz = \int_C (u+iv)(dx+idy) = \int_C u dx - v dy + i \int_C v dx + u dy$$

$$= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dA + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dA = 0$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

Cauchy-Goursat Theorem

$f(z)$: **analytic** in a **simply connected domain** D



for every **simple closed contour** C in D

$$\oint_C f(z) dz = 0$$

$f'(z)$: ~~**continuous**~~ in a simply connected domain D

simple closed curve

a continuously turning tangent

except possibly at a finite number of points

allow a finite number of corners (**not smooth**)

Cauchy's Integral Formula (1)

$f(z)$: **analytic** on and inside simple close curve C



$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$$

the value of $f(z)$
at a point $z = a$ inside C



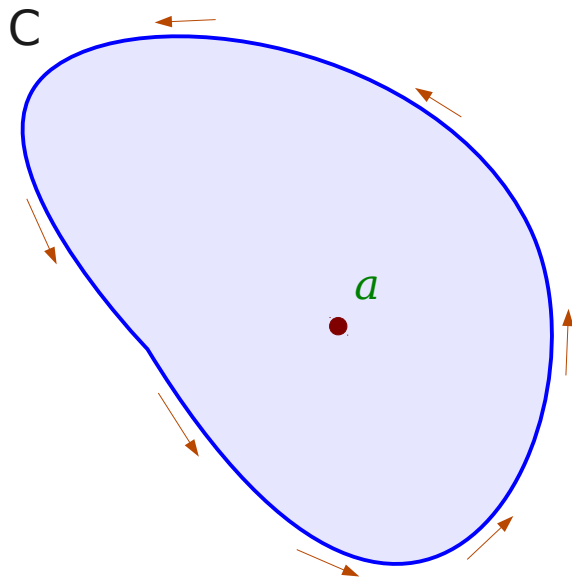
$$f(z) = \frac{1}{2\pi i} \oint \frac{f(w)}{w-z} dw$$

Cauchy's Integral Formula (2)

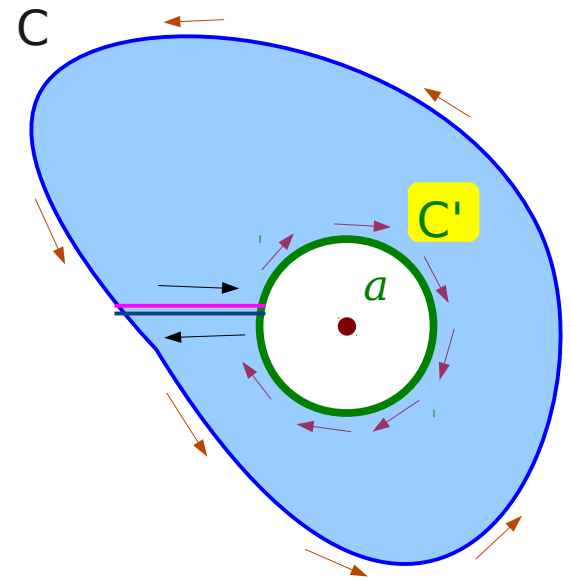
$f(z)$: **analytic** on and inside simple close curve C

➔ $f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$

the value of $f(z)$
at a point $z = a$ inside C



$$\oint_C f(z) dz = 0$$



$$\oint_{\text{ccw } C} \frac{f(z) dz}{z-a} + \oint_{\text{cw } C'} \frac{f(z) dz}{z-a} = 0$$

Cauchy's Integral Formula (3)

$f(z)$: **analytic** on and inside simple close curve C

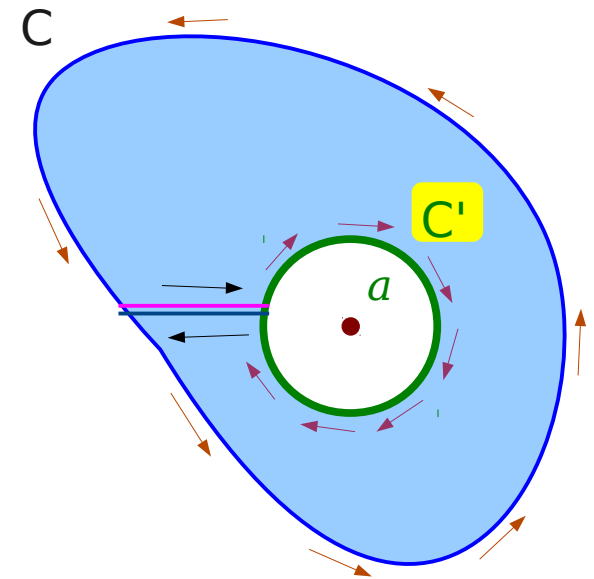


$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$$

the value of $f(z)$
at a point $z = a$ inside C

$$\oint_{\text{ccw } C} \frac{f(z) dz}{z-a} + \oint_{\text{cw } C'} \frac{f(z) dz}{z-a} = 0$$

$$\oint_{\text{ccw } C} \frac{f(z) dz}{z-a} = \oint_{\text{ccw } C'} \frac{f(z) dz}{z-a}$$



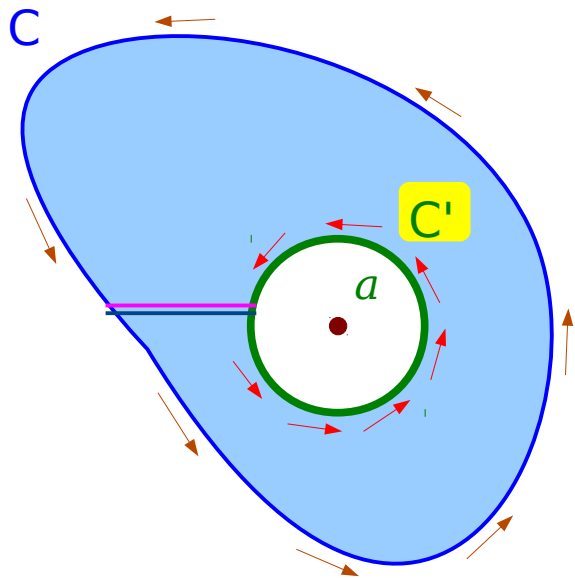
Cauchy's Integral Formula (4)

$$\oint_{\text{ccw } C} \frac{f(z) dz}{z-a} = \oint_{\text{ccw } C'} \frac{f(z) dz}{z-a}$$

along C' $z - a = \rho e^{i\theta}$

$$= 2\pi i f(a)$$

as $z \rightarrow a \Rightarrow \rho \rightarrow 0$

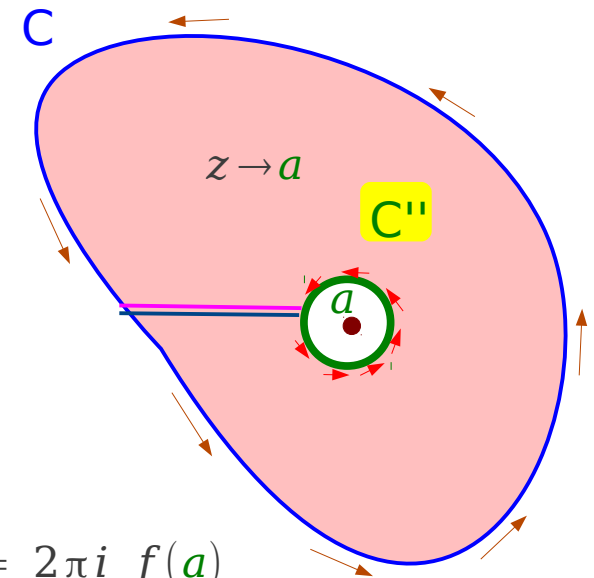


$$z = a - \rho e^{i\theta}$$

$$dz = i\rho e^{i\theta} d\theta$$

$$\frac{dz}{z-a} = \frac{i\rho e^{i\theta} d\theta}{\rho e^{i\theta}}$$

$$\oint_{\text{ccw } C} \frac{f(z) dz}{z-a} = \int_0^{2\pi} f(z) i d\theta = 2\pi i f(a)$$



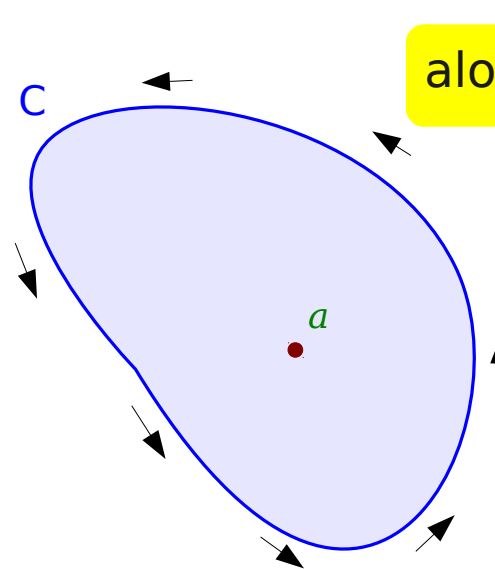
Cauchy's Integral Formula (5)

$$\frac{dz}{(z-a)^2} = \frac{i\rho e^{i\theta} d\theta}{(\rho e^{i\theta})^2}$$

$$\oint_{\text{ccw } C} \frac{f(z) dz}{(z-a)^2} = \int_0^{2\pi} \frac{f(z)i}{\rho e^{i\theta}} d\theta$$

$$= \int_0^{2\pi} \frac{f(z)}{\rho} i e^{-i\theta} d\theta = \left[-\frac{f(z)}{\rho} e^{-i\theta} \right]_0^{2\pi}$$

$$= -\frac{f(z)}{\rho} (e^{-i2\pi} - e^{-i0}) = 0$$



along C' $z - a = \rho e^{i\theta}$

$$z = a - \rho e^{i\theta}$$

$$dz = i\rho e^{i\theta} d\theta$$

$$dz = i\rho e^{i\theta} d\theta$$

$$\oint_{\text{ccw } C} f(z) dz = \int_0^{2\pi} f(z) i\rho e^{i\theta} d\theta$$

$$= \left[f(z) \rho e^{i\theta} \right]_0^{2\pi}$$

$$= f(z) \rho (e^{-i2\pi} - e^{-i0}) = 0$$

$$(z-a) dz = \rho e^{i\theta} i\rho e^{i\theta} d\theta$$

$$\oint_{\text{ccw } C} (z-a) f(z) dz = \int_0^{2\pi} f(z) i(\rho e^{i\theta})^2 d\theta$$

$$= \int_0^{2\pi} f(z) \rho^2 i e^{i2\theta} d\theta = \left[f(z) \frac{\rho}{2} e^{i2\theta} \right]_0^{2\pi}$$

$$= f(z) \frac{\rho}{2} (e^{-i4\pi} - e^{-i0}) = 0$$

References

- [1] <http://en.wikipedia.org/>
- [2] <http://planetmath.org/>
- [3] M.L. Boas, “Mathematical Methods in the Physical Sciences”