Row Reduction

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09/03/2012

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Linear Equations

$$
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= b_m
\end{aligned}
$$

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ \vdots \\ x_n
\end{pmatrix}
\begin{matrix} = \\ = \\ \\ = \end{matrix}
\begin{pmatrix}
b_1 \\ b_2 \\ \vdots \\ b_m
\end{pmatrix}
$$

Linear Equations

$$
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= b_m
\end{aligned}
$$

$$
\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
\begin{matrix} = \\ = \\ \\ = \end{matrix}
\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}
$$

$$
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} b_1 \end{bmatrix}
\qquad \sum_{j=1}^{n} a_{1j} \cdot x_j = b_1
$$

$$
\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} b_2 \end{bmatrix}
\qquad \sum_{j=1}^{n} a_{2j} \cdot x_j = b_2
$$

$$
\begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b_m
\qquad \sum_{j=1}^{n} a_{mj} \cdot x_j = b_m
$$

Example

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

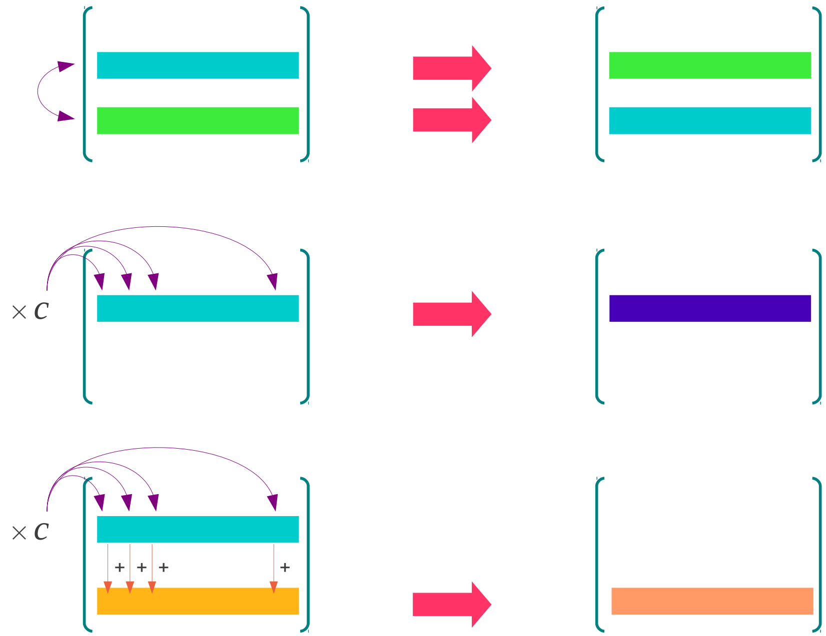
$$\begin{aligned}
2\,x_1 + 1\,x_2 - 1\,x_3 &= +8 \\
-3\,x_1 - 1\,x_2 + 2\,x_3 &= -11 \\
-2\,x_1 + 1\,x_2 + 2\,x_3 &= -3
\end{aligned}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\begin{pmatrix} +2 & +1 & -1 \\ -3 & -1 & +2 \\ -2 & +1 & +2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_n \end{pmatrix} = \begin{pmatrix} +8 \\ -11 \\ -3 \end{pmatrix}$$

Gauss-Jordan Elimination

$$\begin{pmatrix} +2 & +1 & -1 \\ -3 & -1 & +2 \\ -2 & +1 & +2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} +8 \\ -11 \\ -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} * \\ * \\ * \end{pmatrix}$$



Gauss-Jordan Elimination – Step 1

$$+2x_1 + x_2 - x_3 = 8 \qquad (L_1)$$
$$-3x_1 - x_2 + 2x_3 = -11 \qquad (L_2)$$
$$-2x_1 + x_2 + 2x_3 = -3 \qquad (L_3)$$

$$\left[\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right]$$

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = 4 \qquad \left(\tfrac{1}{2} \times L_1\right)$$

+2/2 +1/2 –1/2 +8/2

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = 4 \qquad \left(\tfrac{1}{2} \times L_1\right)$$
$$-3x_1 - x_2 + 2x_3 = -11 \qquad (L_2)$$
$$-2x_1 + x_2 + 2x_3 = -3 \qquad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right]$$

Gauss-Jordan Elimination – Step 2

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \quad (L_1)$$
$$-3x_1 - x_2 + 2x_3 = -11 \quad (L_2)$$
$$-2x_1 + x_2 + 2x_3 = -3 \quad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right)$$

$$+3x_1 + \tfrac{3}{2}x_2 - \tfrac{3}{2}x_3 = +12 \quad (3 \times L_1)$$
$$-3x_1 - x_2 + 2x_3 = -11 \quad (L_2)$$

+3	+3/2	–3/2	+12
–3	–1	+2	–11

$$+2x_1 + \tfrac{2}{2}x_2 - \tfrac{2}{2}x_3 = +8 \quad (2 \times L_1)$$
$$-2x_1 + x_2 + 2x_3 = -3 \quad (L_3)$$

+2	+2/2	–2/2	+8
–2	+1	+2	–3

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \quad (L_1)$$
$$0x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 = +1 \quad (3 \times L_1 + L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \quad (2 \times L_1 + L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array}\right)$$

Gauss-Jordan Elimination – Step 3

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 = +1 \qquad (L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array}\right)$$

$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (2 \times L_2)$$

$$\begin{array}{cccc} 0 & +1 & +1 & +2 \end{array}$$

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (2 \times L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right)$$

Gauss-Jordan Elimination – Step 4

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right)$$

$$0x_1 - 2x_2 - 2x_3 = -4 \qquad (-2 \times L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (L_3)$$

0	–2	–2	–4
0	+2	+1	+5

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 0x_2 - 1x_3 = +1 \qquad (-2 \times L_2 + L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array}\right)$$

Gauss-Jordan Elimination – Step 5

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 0x_2 - 1x_3 = +1 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array}\right)$$

$$0x_1 - 0x_2 + 1x_3 = -1 \qquad (-1 \times L_3)$$

$$\begin{array}{cccc} 0 & 0 & +1 & -1 \end{array}$$

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 0x_2 + 1x_3 = -1 \qquad (-1 \times L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

Forward Phase

$$\left(\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow$$

Forward Phase - Gaussian Elimination

Gauss-Jordan Elimination – Step 6

$$+1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 \quad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \quad (L_2)$$
$$0x_1 + 0x_2 + 1x_3 = -1 \quad (L_3)$$

+1	+1/2	–1/2		+4
0	+1	+1		+2
0	0	+1		–1

$$0x_1 + 0x_2 + \frac{1}{2}x_3 = -\frac{1}{2} \quad \left(+\frac{1}{2} \times L_3\right)$$
$$+1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 \quad (L_1)$$

0	0	+1/2	–1/2
+1	+1/2	–1/2	+4

$$0x_1 + 0x_2 - 1x_3 = +1 \quad (-1 \times L_3)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \quad (L_2)$$

0	0	–1	+1
0	+1	+1	+2

$$+1x_1 + \frac{1}{2}x_2 + 0x_3 = +\frac{7}{2} \quad \left(+\frac{1}{2} \times L_3 + L_1\right)$$
$$0x_1 + 1x_2 + 0x_3 = +3 \quad (-1 \times L_3 + L_2)$$
$$0x_1 + 0x_2 + 1x_3 = -1 \quad (L_3)$$

+1	+1/2	0		+7/2
0	+1	0		+3
0	0	+1		–1

Gauss-Jordan Elimination – Step 7

$$+1x_1 + \tfrac{1}{2}x_2 + 0x_3 = +\tfrac{7}{2} \qquad (L_1)$$
$$0x_1 + 1x_2 + 0x_3 = +3 \qquad (L_2)$$
$$0x_1 + 0x_2 + 1x_3 = -1 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & \boxed{+1/2} & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

$$0x_1 - \tfrac{1}{2}x_2 + 0x_3 = -\tfrac{3}{2} \qquad \boxed{\left(-\tfrac{1}{2} \times L_2\right)}$$
$$+1x_1 + 0x_2 - 0x_3 = +2 \qquad (L_1)$$

0	–1/2	0	–3/2
+1	+1/2	0	+7/2

$$+1x_1 + 0x_2 - 0x_3 = +2 \qquad (+1 \times L_3 + L_1)$$
$$0x_1 + 1x_2 + 0x_3 = +3 \qquad (L_2)$$
$$0x_1 + 0x_2 + 1x_3 = -1 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & \boxed{0} & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

Backward Phase

$$\begin{pmatrix} +1 & +1/2 & -1/2 & | & +4 \\ 0 & +1 & +1 & | & +2 \\ 0 & 0 & +1 & | & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} +1 & +1/2 & 0 & | & +7/2 \\ 0 & +1 & 0 & | & +3 \\ 0 & 0 & +1 & | & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} +1 & 0 & 0 & | & +2 \\ 0 & +1 & 0 & | & +3 \\ 0 & 0 & +1 & | & -1 \end{pmatrix} \Rightarrow$$

Gauss-Jordan Elimination

Forward Phase - Gaussian Elimination

$$\left(\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow$$

Backward Phase

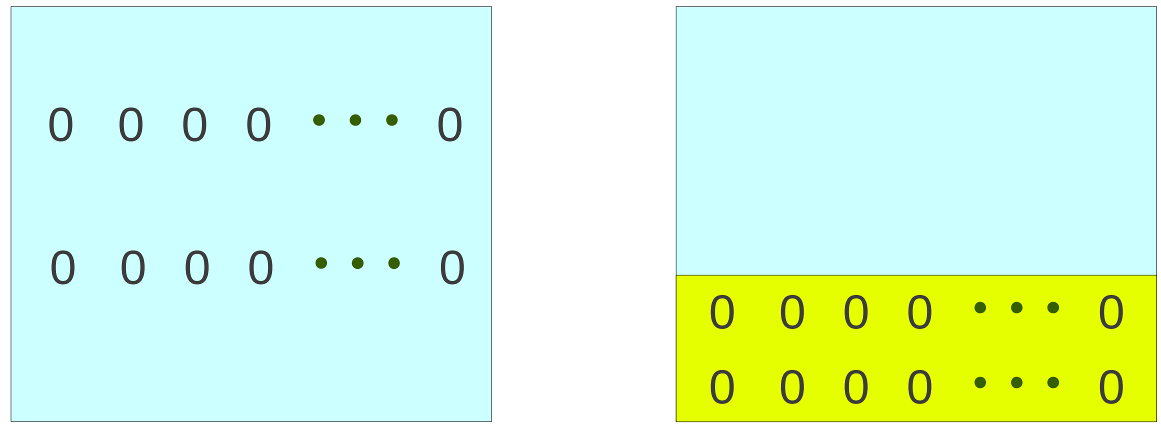
$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & 0 & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow$$

Echelon Forms (1)

zero rows	⇒	Should be grouped at the bottom
non-zero row	⇒	A leading **1** The 1st non-zero element should be one
Any successive non-zero rows	⇒	The leading **1** of the lower row should be farther to the **right** than the leading **1** of the higher row

Echelon Forms (2)

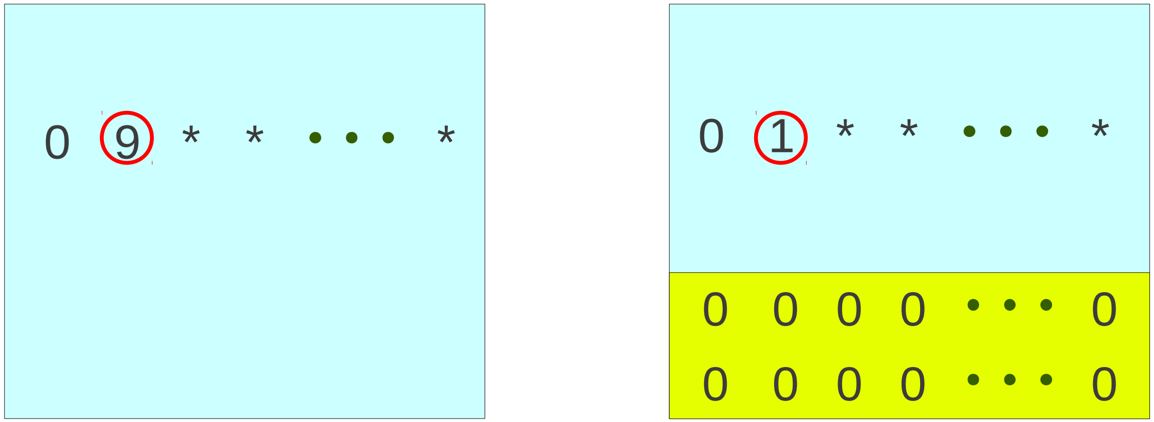
zero rows → Should be grouped at the bottom



Echelon Forms (3)

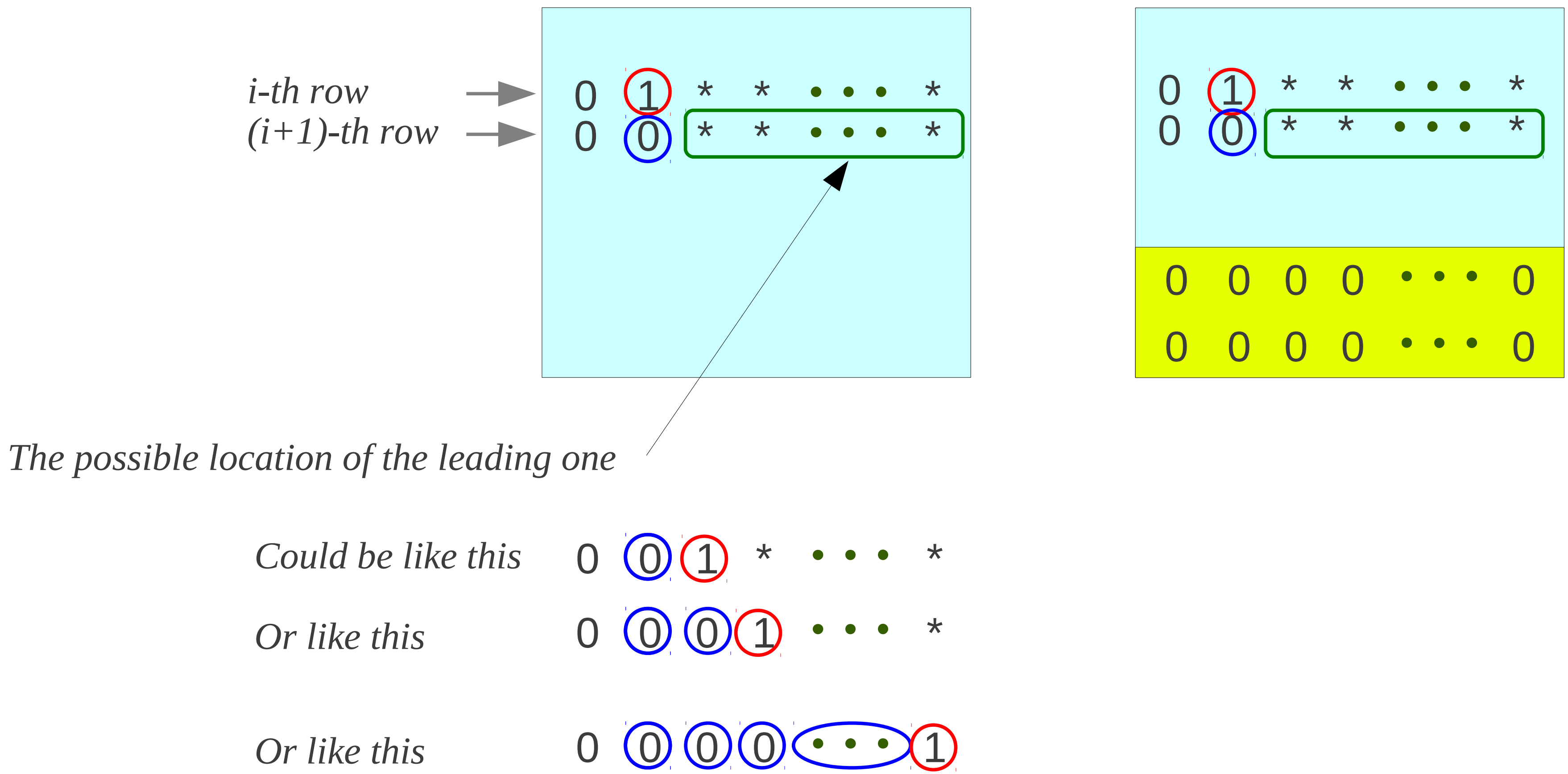
non-zero row → A leading one

The 1st non-zero element should be one



Echelon Forms (3)

Any successive non-zero rows → The leading **1** of the lower row should be farther to the **right** than the leading **1** of the higher row

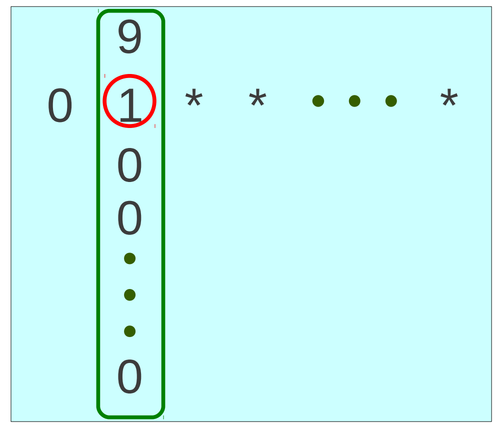


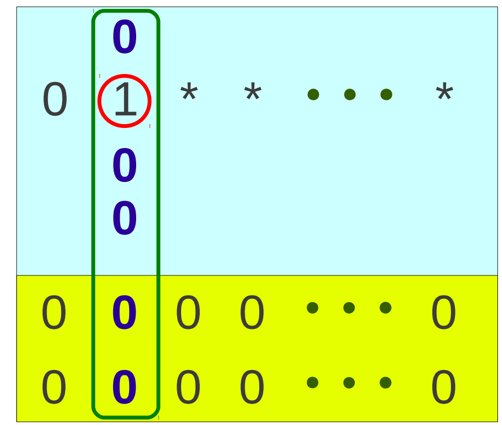
Reduced Echelon Forms

zero rows →	Should be grouped at the bottom
non-zero row →	A leading **1** The 1st non-zero element should be one
Any successive non-zero rows →	The leading **1** of the lower row should be farther to the **right** than the leading **1** of the higher row
Any column that contains a leading **1** →	All other elements except the leading one are **all zeros**

Reduced Echelon Forms

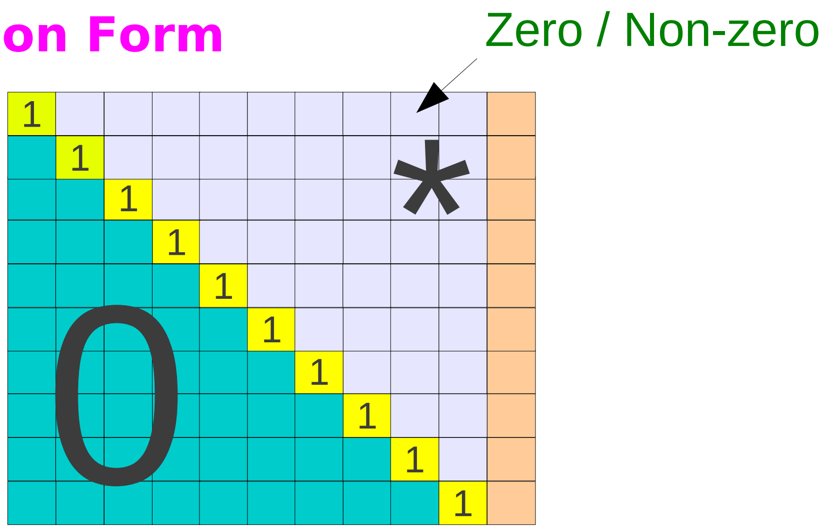
Any column that contains a leading one → All other elements except the leading one are **all zeros**

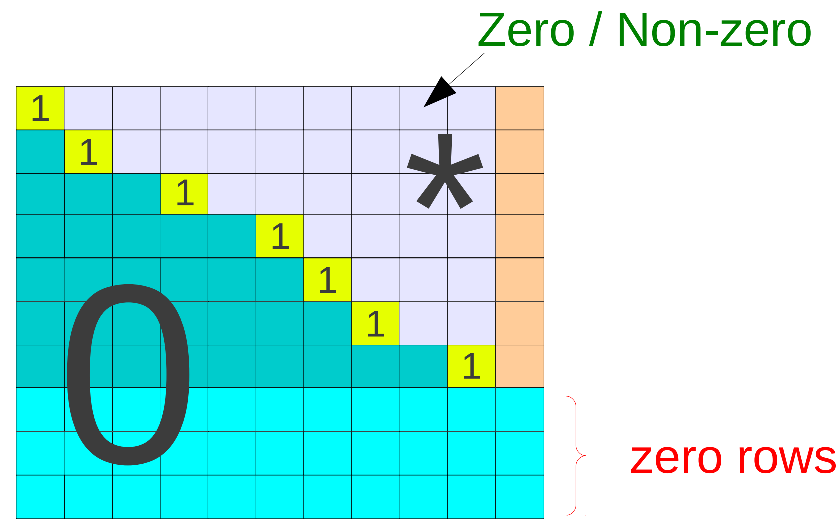




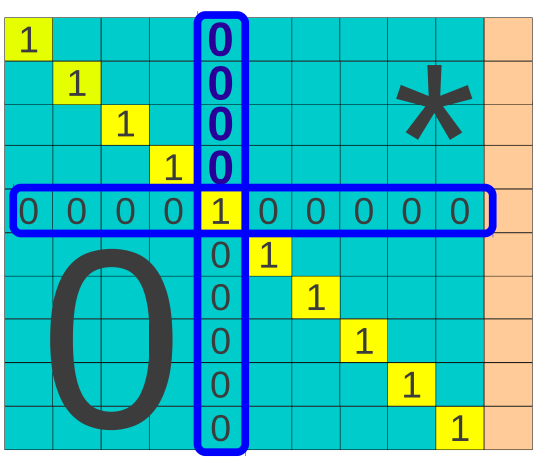
Examples

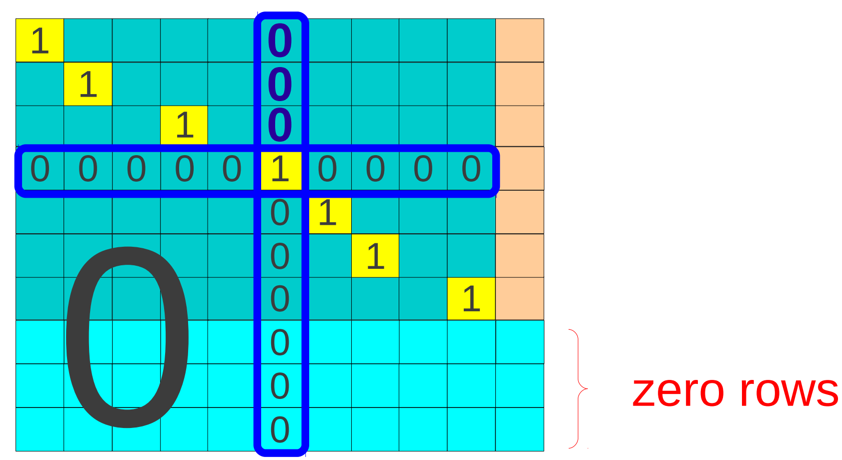
Echelon Form





Reduced Echelon Form



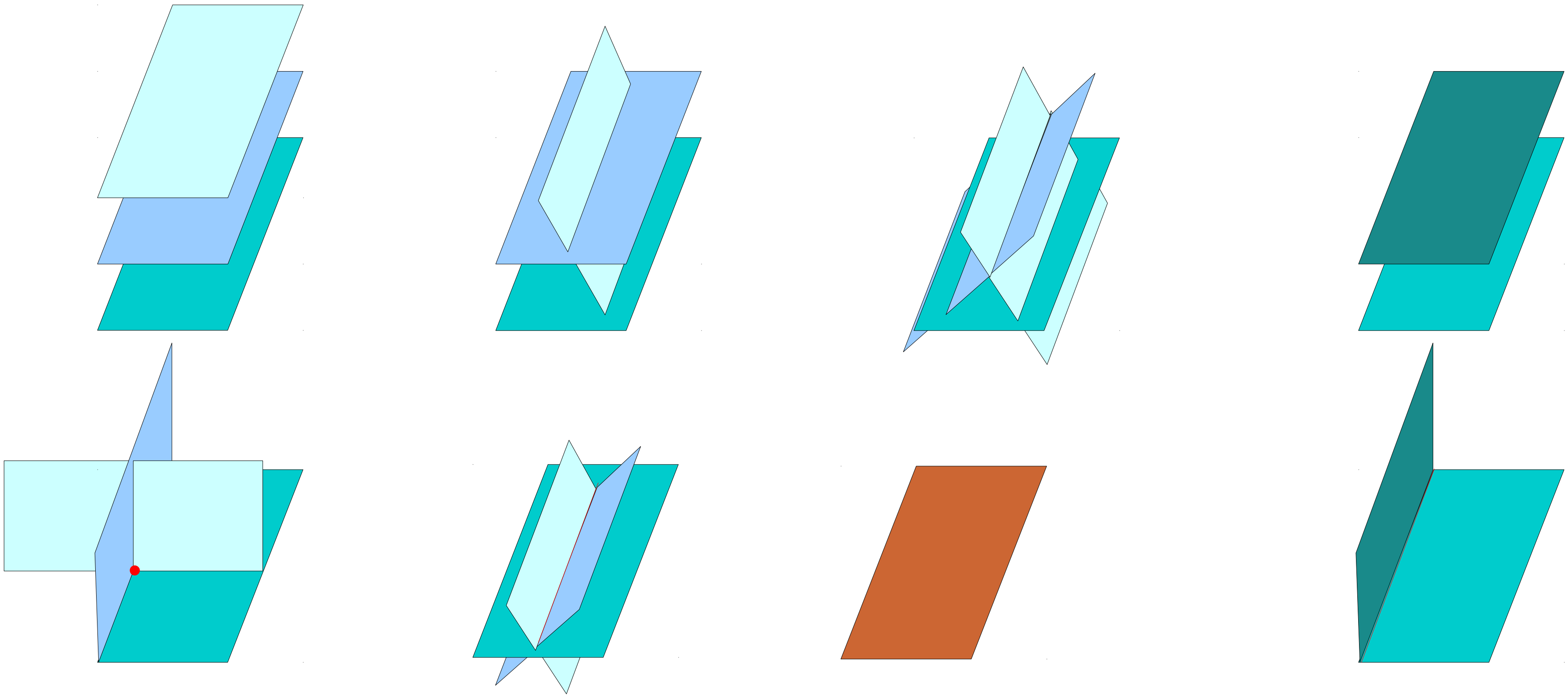


Linear Systems of 3 Unknowns

(Eq 1) $a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$

(Eq 2) $a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$

(Eq 3) $a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3$



Leading and Free Variables

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$$

$$0 = 1$$

$$1 \cdot x_1 + 3 \cdot x_3 = -1$$
$$1 \cdot x_2 - 4 \cdot x_3 = 2$$

$$1 \cdot x_1 - 5 \cdot x_2 + 1 \cdot x_3 = 4$$

with a leading 1

leading variables

Other remaining varaible

free variables

Free Variables as Parameters

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$$

$$0 = 1$$

$$1 \cdot x_1 + 3 \cdot x_3 = -1$$
$$1 \cdot x_2 - 4 \cdot x_3 = 2$$

$$1 \cdot x_1 - 5 \cdot x_2 + 1 \cdot x_3 = 4$$

Solve for a leading variable

$$x_1 = -1 - 3 \cdot x_3$$
$$x_2 = 2 + 4 \cdot x_3$$

$$x_1 = 4 + 5 \cdot x_2 - 1 \cdot x_3$$

Treat a free variable
as a parameter

$$x_3 = t$$

$$x_2 = s \qquad x_3 = t$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

$$\begin{cases} x_1 = 4 + 5 \cdot x_2 - 1 \cdot x_3 \\ x_2 = s \\ x_3 = t \end{cases}$$

Free Variables as Parameters

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$$

$$0 = 1$$

$$1 \cdot x_1 + 3 \cdot x_3 = -1$$
$$1 \cdot x_2 - 4 \cdot x_3 = 2$$

$$1 \cdot x_1 - 5 \cdot x_2 + 1 \cdot x_3 = 4$$

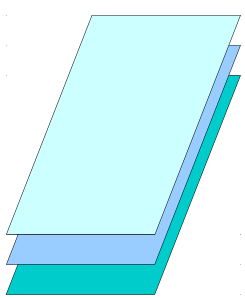
$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

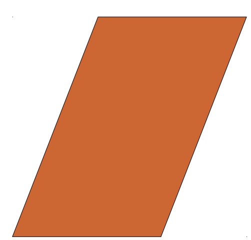
$x_3 = t$ ⇐ free variable

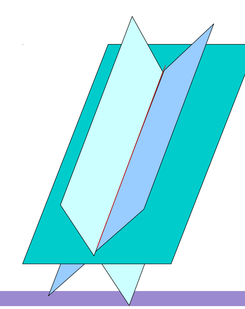
$$\begin{cases} x_1 = 4 + 5 \cdot x_2 - 1 \cdot x_3 \\ x_2 = s \\ x_3 = t \end{cases}$$

$x_2 = s$ ⇐ free variable

$x_3 = t$ ⇐ free variable







Consistent Linear System

A linear system with at least one solution

➡ A Consistent Linear System

A linear system with no solutions

➡ A Inconsistent Linear System

General Solution

A linear system with infinitely many solutions

Solve for a leading variable

Treat a free variable as a parameter

➡ A set of parametric equations

All solutions can be obtained
by assigning numerical values to those parmeters

➡ Called a general solution

Homogeneous System

$$
\begin{array}{ccccccccc}
a_{11} x_1 & + & a_{12} x_2 & + & \cdots & + & a_{1n} x_n & = & 0 \\
a_{21} x_1 & + & a_{22} x_2 & + & \cdots & + & a_{2n} x_n & = & 0 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1} x_1 & + & a_{m2} x_2 & + & \cdots & + & a_{mn} x_n & = & 0
\end{array}
$$

All constant terms are zero

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ \vdots \\ x_n
\end{pmatrix}
=
\begin{pmatrix}
0 \\ 0 \\ \vdots \\ 0
\end{pmatrix}
$$

All constant terms are zero

Solutions of a Homogeneous System

All homogeneous system passes through the origin

The homogeneous system has

* only the trivial solution

* <u>many solutions</u> in addition to the trivial solution

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
$$

$$
\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
\begin{matrix} = \\ = \\ \\ = \end{matrix}
\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
$$

Trivial Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{matrix} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = 0 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = 0 \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = 0 \end{matrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

satisfies all homogeneous equation

All homogeneous system passes through the origin

Augmented Matrix

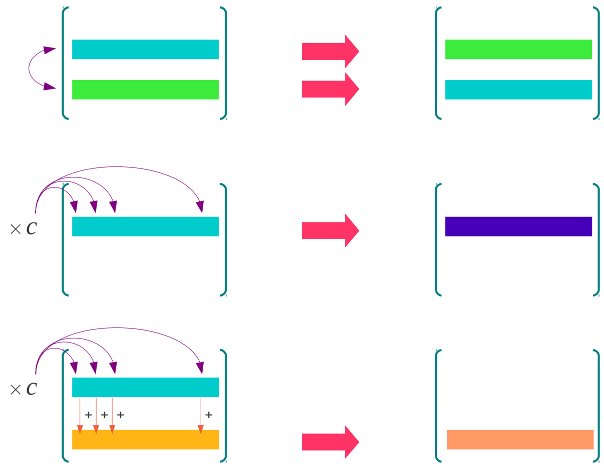
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}$$

Augmented matrix of a homogeneous system

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right)$$

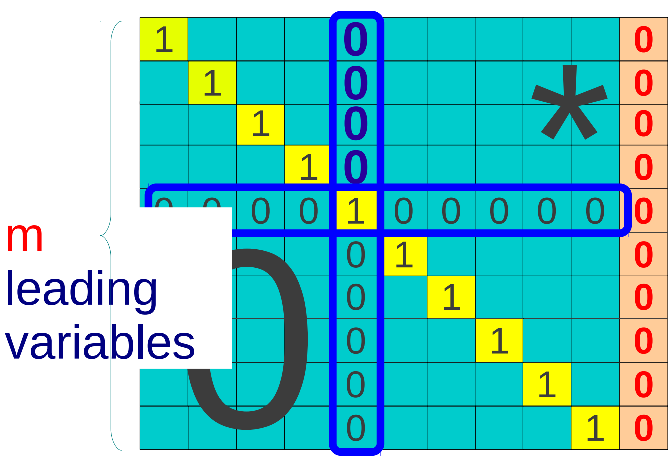
Reduced Row Echelon Form

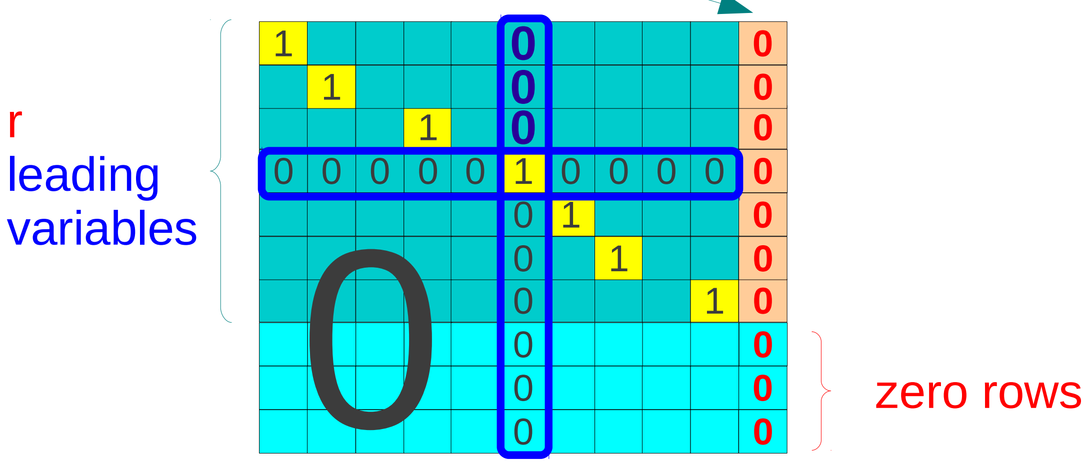


Elementary row operations do not alter the zero column of in a matrix

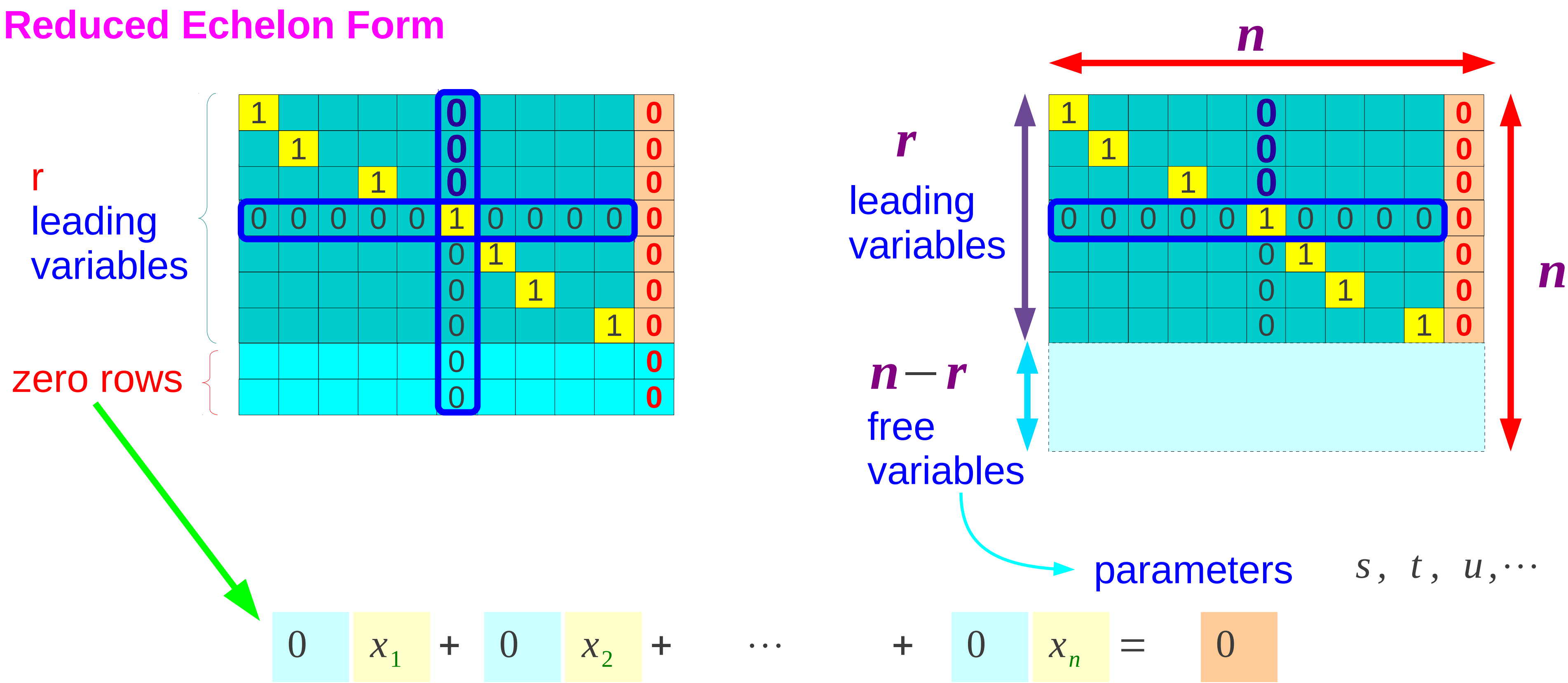
The augmented zero column is preserved in the reduced echelon form of a homogeneous system

Reduced Echelon Form





Free Variable Theroem



A homogeneous linear system with ***n* unknowns**

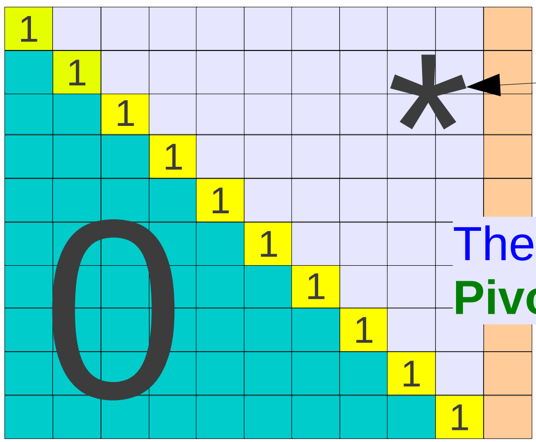
If the reduced row echelon form of its augmented matrix has

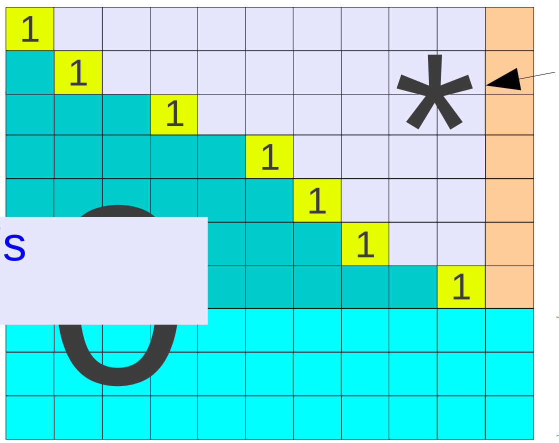
r non-zero rows ➡ ***n* – *r*** free variables ➡ *infinitely many solutions*

Pivot Positions

Echelon Form → **Not unique**

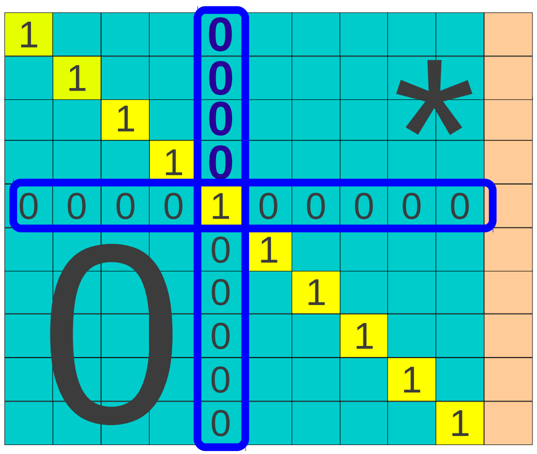
Depend on the sequence of elementary row operations

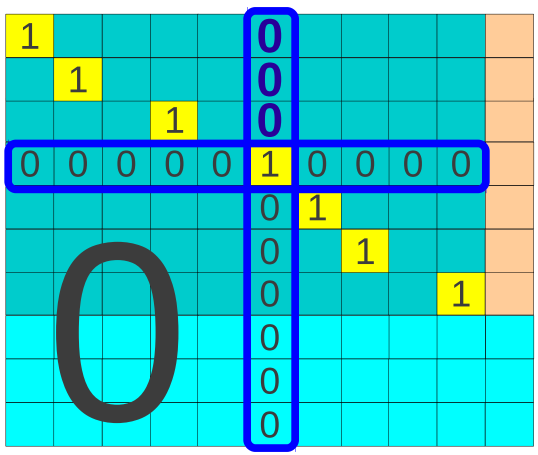




The position of leading 1's
Pivot position is unique

Reduced Echelon Form → **Unique**





Pulse

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Pulse

References

[1] http://en.wikipedia.org/
[2] Anton & Busby, “Contemporary Linear Algebra”
[3] Anton & Rorres, “Elementary Linear Algebra”