Complex Functions (1A)

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Analytic Functions

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$$

$$\Delta f = f(z+\Delta z) - f(z)$$

$$\Delta z = \Delta x + i \Delta y$$

$f(z)$: **analytic** in a region ⟺

$f(z)$ has a (unique) derivative at every point of the region

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$$

$f(z)$: **analytic** at a point $z = a$ ⟺

$f(z)$ has a (unique) derivative at every point of some small circle about $z = a$

Singular Point

Regular point of $f(z)$ ⟺

a point at which $f(z)$ is analytic

Singular point of $f(z)$ ⟺

a point at which $f(z)$ is not analytic

Isolated Singular point of $f(z)$ ⟺

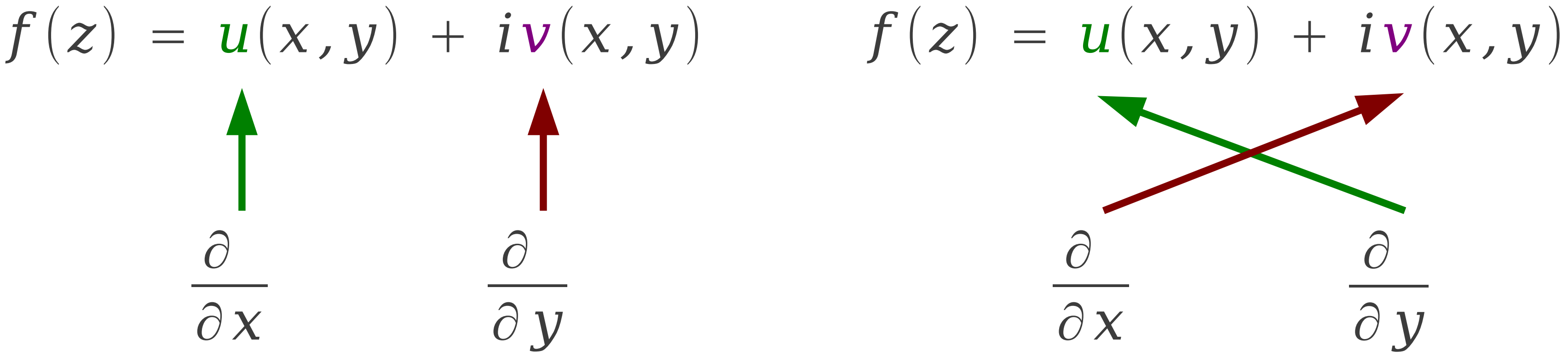
a point at which $f(z)$ is analytic everywhere

else inside some small circle about the singular point

Cauchy-Riemann Condition

$f(z) = u(x,y) + i v(x,y)$: **analytic** in a region

➡ in that region $\quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \qquad \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$



$$f(z) = u(x,y) + i v(x,y) \qquad \frac{\partial}{\partial x} \qquad \frac{\partial}{\partial y}$$

$$f(z) = u(x,y) + i v(x,y) \qquad \frac{\partial}{\partial x} \qquad \frac{\partial}{\partial y}$$

Analytic

$$f(z) = u(x,y) + i v(x,y)$$

$$u(x,y),\ v(x,y),\ \frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y},\ \frac{\partial v}{\partial x},\ \frac{\partial v}{\partial y}$$: continuous

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

➡ $$f(z) = u(x,y) + i v(x,y)$$

: **analytic** at all points inside a region

not necessarily on the boundary

Derivatives

$f(z) = u(x,y) + i v(x,y)$: **analytic** in a region R

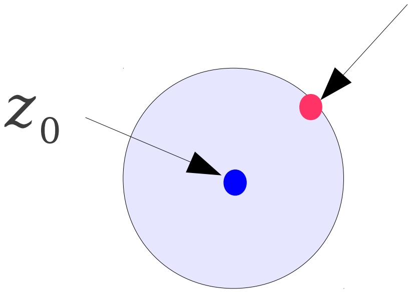
- derivatives of all orders at points inside region

$$f'(z_0),\ f''(z_0),\ f^{(3)}(z_0),\ f^{(4)}(z_0),\ f^{(5)}(z_0),\ \cdots$$

- Taylor series expansion about any point z_0 inside the region

The power series converges inside the circle about z_0

This circle extends to the nearest singular point



Laplace Equation

$f(z) = u(x,y) + iv(x,y)$: **analytic** in a region R

➡ $u(x,y),\ v(x,y)$ satisfy Laplace's equation in the region

harmonic functions

$u(x,y),\ v(x,y)$ satisfy Laplace's equation in simply connected region

➡ Real / imaginary part of an analytic function $f(z)$

Cauchy's Theorem

$f(z)$: **analytic** on and inside C ➡ $\oint_{around\,C} f(z)\,dz = 0$

simple closed curve

a continuously turning tangent
except possibly at a finite number of points

allow a finite number of corners (not smooth)

Cauchy's Integral Formula

$f(z)$: **analytic** on and inside simple close curve C

➡ $$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$$

the value of $f(z)$

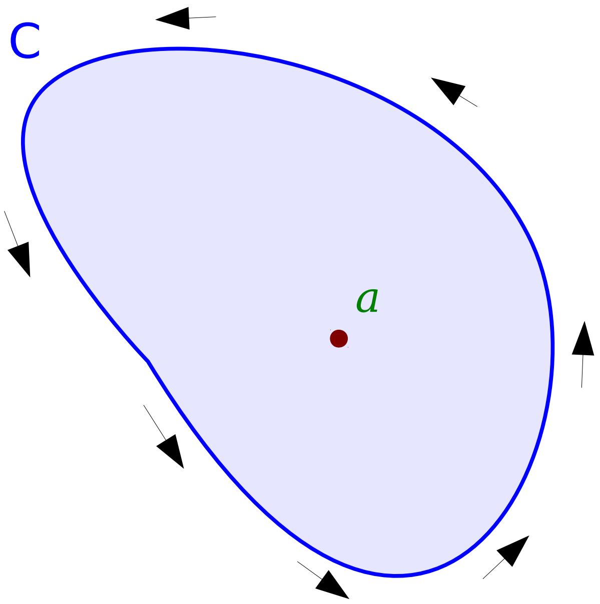
at a point $z = a$ inside C

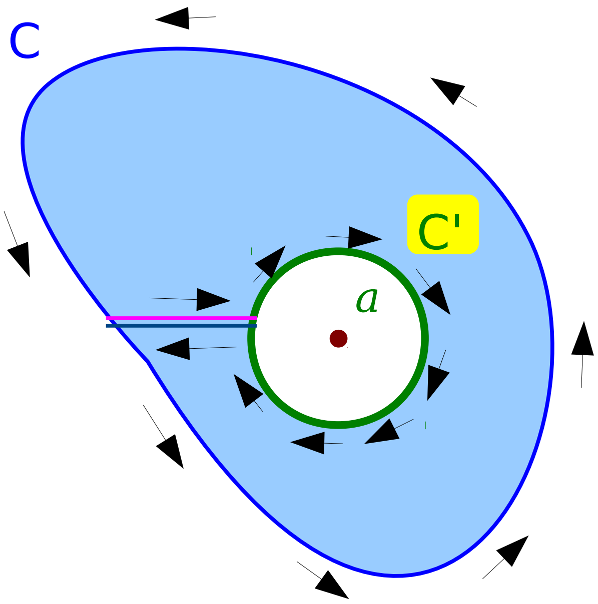
$$f(z) = \frac{1}{2\pi i} \oint \frac{f(w)}{w-z} dw$$

Cauchy's Integral Formula

$f(z)$: **analytic** on and inside simple close curve C

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}\,dz$$





$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} = \oint_{ccw\ C'} \frac{f(z)\,dz}{z-a}$$

$$\oint_C f(z)\,dz = 0$$

$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} + \oint_{cw\ C'} \frac{f(z)\,dz}{z-a} = 0$$

Cauchy's Integral Formula

$f(z)$: **analytic** on and inside simple close curve C

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}dz$$

along C' $z - a = \rho e^{i\theta}$

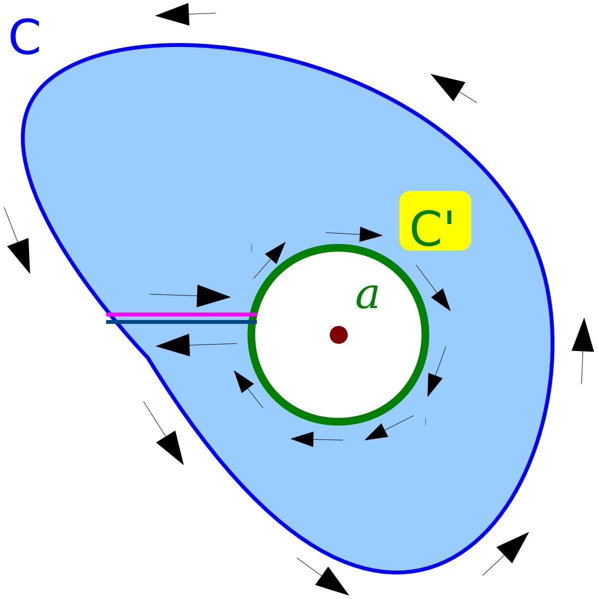
$z = a - \rho e^{i\theta}$

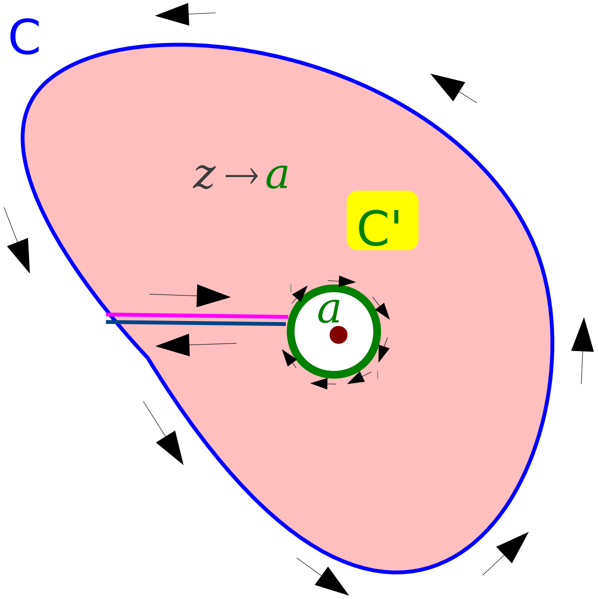
$dz = i\rho e^{i\theta} d\theta$

$$\frac{dz}{z-a} = \frac{i\rho e^{i\theta} d\theta}{\rho e^{i\theta}}$$

$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} = \int_0^{2\pi} f(z) i\, d\theta = 2\pi i\ f(a)$$

as $z \to a$ ➡ $\rho \to 0$





$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} = \oint_{ccw\ C'} \frac{f(z)\,dz}{z-a}$$

$$= 2\pi i\ f(a)$$

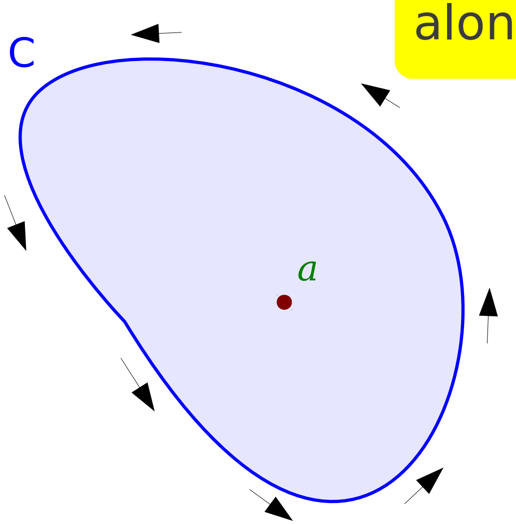
Cauchy's Integral Formula

$$\frac{dz}{(z-a)^2} = \frac{i\rho e^{i\theta}\,d\theta}{(\rho e^{i\theta})^2}$$

$$\boxed{\oint_{ccw\ C} \frac{f(z)\,dz}{(z-a)^2}} = \int_0^{2\pi} \frac{f(z)i}{\rho e^{i\theta}}\,d\theta$$

$$= \int_0^{2\pi} \frac{f(z)}{\rho} i e^{-i\theta}\,d\theta = \left[-\frac{f(z)}{\rho} e^{-i\theta}\right]_0^{2\pi}$$

$$= -\frac{f(z)}{\rho}(e^{-i2\pi} - e^{-i0}) = 0$$



along C' $z - a = \rho e^{i\theta}$

$$z = a - \rho e^{i\theta}$$

$$dz = i\rho e^{i\theta}\,d\theta$$

$$dz = i\rho e^{i\theta}\,d\theta$$

$$\boxed{\oint_{ccw\ C} f(z)\,dz} = \int_0^{2\pi} f(z) i\rho e^{i\theta}\,d\theta$$

$$= \left[f(z)\rho e^{i\theta}\right]_0^{2\pi}$$

$$= f(z)\rho(e^{-i2\pi} - e^{-i0}) = 0$$

$$(z-a)\,dz = \rho e^{i\theta} i\rho e^{i\theta}\,d\theta$$

$$\boxed{\oint_{ccw\ C} (z-a) f(z)\,dz} = \int_0^{2\pi} f(z) i(\rho e^{i\theta})^2\,d\theta$$

$$= \int_0^{2\pi} f(z)\rho^2 i e^{i2\theta}\,d\theta = \left[f(z)\frac{\rho}{2} e^{i2\theta}\right]_0^{2\pi}$$

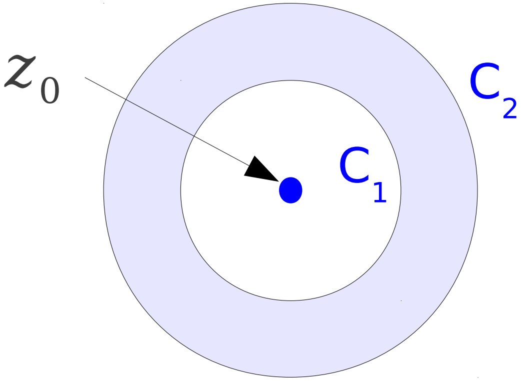
$$= f(z)\frac{\rho}{2}(e^{-i4\pi} - e^{-i0}) = 0$$

Laurent's Theorem

$f(z)$: **analytic** in the region R

between circles C_1, C_2

centered at z_0



$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

Principal part

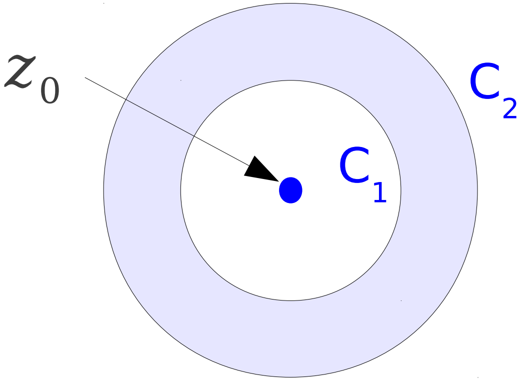
: **convergent** in the region R

Laurent's Theorem - Region of Convergence

$f(z)$: **analytic** in the region R

between circles C_1, C_2

centered at z_0



$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

For this “a” series to converge,
the ROC must be in the form

$$|z-z_0| < const$$ ⇒ inside C_2

$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

Principal part

For this “b” series to converge,
the ROC must be in the form

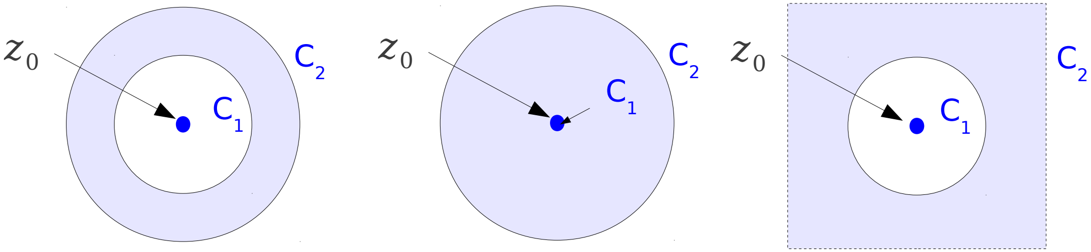
$$\left|\frac{1}{z-z_0}\right| < const$$ ⇒ outside C_1

Laurent's Theorem - Coefficients

$f(z)$: **analytic** in the region R

between circles C_1, C_2

centered at z_0



$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

: **convergent** in the region R

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

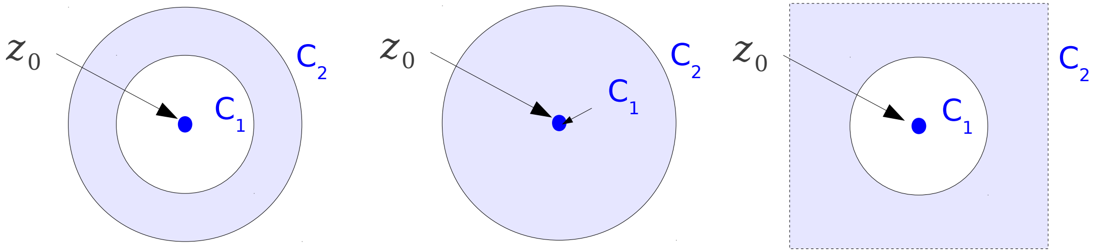
$$b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}}$$

Laurent's Theorem – Some Points

$f(z)$: **analytic** in the region R

between circles C_1, C_2

centered at z_0



$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

regular point z_0

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n}$$

pole of order n z_0

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{(z-z_0)}$$

simple pole z_0

b_1 *residue of* $f(z)$

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \cdots$$

essential singularity z_0

Residue Theorem (1)

$f(z)$: **analytic** on and inside C (no singular point)

$$\oint_C f(z)\,dz = 0$$

$f(z)$: **analytic** on and inside C except z_0

$$\oint_C f(z)\,dz = 2\pi i \cdot \sum \textit{the residues of } f(z) \textit{ inside } C$$

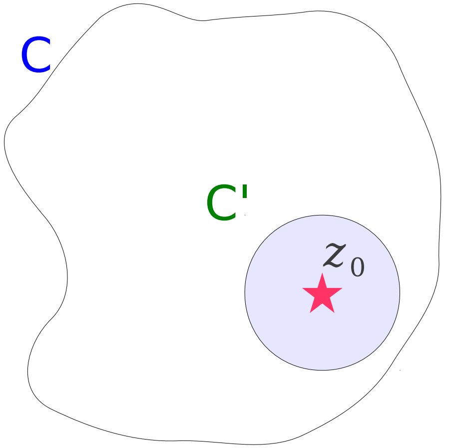
z_0 Isolated singular point

The integral around C is in the counterclockwise direction

Residue Theorem (2)

$f(z)$: **analytic** on and inside C except z_0

z_0 Isolated singular point



$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \cdots$$

$$\oint_C a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \, dz = 0$$

analytic on and inside C

along C' $z = z_0 + \rho e^{i\theta}$

$$\oint_{C'} \frac{b_1}{(z-z_0)} dz = b_1 \int_0^{2\pi} \frac{\rho i e^{i\theta} d\theta}{\rho e^{i\theta}} = b_1 \int_0^{2\pi} i \, d\theta = 2\pi i b_1$$

$$\oint_{C'} \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \cdots \, dz = 0$$

$$\int_0^{2\pi} e^{ik\theta} d\theta = \left[\frac{e^{ik\theta}}{ik} \right]_0^{2\pi} = 0$$

Finding Residues (1)

$$\oint_C f(z)\,dz = 2\pi i \cdot \sum \textit{the residues of } f(z) \textit{ inside } C$$

The integral around C is in the counterclockwise direction

Methods of Finding Residues

Laurent Series: b_1 $1/(z-z_0)$

Simple Pole : $f(z)\cdot(z-z_0)$

Multiple Pole : $f(z)\cdot(z-z_0)^m$

Finding Residues (2)

$$\oint_C f(z)\,dz = 2\pi i \cdot \sum \textit{the residues of } f(z) \textit{ inside } C$$

The integral around C is in the counterclockwise direction

Laurent Series: $b_1 \qquad\qquad 1/(z-z_0)$

Simple Pole : $f(z)\cdot(z-z_0)$

Multiple Pole : $f(z)\cdot(z-z_0)^m$

$$R(z_0) = \lim_{z \to z_0} (z-z_0) f(z)$$

$$R(z_0) = \frac{g(z_0)}{h'(z_0)} \quad \Leftarrow \quad f(z) = \frac{g(z)}{h(z)} \qquad \begin{array}{l} g(z_0) \neq 0 \\ h'(z_0) \neq 0 \quad h(z_0) = 0 \end{array}$$

$$R(z_0) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)$$

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”